When a system of quadratic equations has a solution

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Joint work with Mark Rudelson

A. Barvinok and M. Rudelson, When a system of real quadratic equations has a solution. *Adv. Math.* **403** (2022), Paper No. 108391, 38 pp.

Solving systems of polynomial equations

Given a system of real polynomial equations

$$p_i(x_1, \ldots, x_n) = 0 \quad \text{for} \quad i = 1, \ldots, m,$$

how hard is it to

a) decide if there is a solution

b) if there is a solution, to find one

c) describe the set of all solutions?

Answer: Generally speaking, pretty hard.
A good reference: S. Basu, R. Pollack, and M.-F. Roy, Algorithms in Real Algebraic Geometry. Second edition, Algorithms and Computation in Mathematics, **10** Springer-Verlag, Berlin, 2006. x+662.

Solving systems of polynomial equations

Two main parameters: the number n of variables and the largest degree d of the equation. Any number of equations can be reduced to one by doubling the degree:

$$p_i(x_1, \ldots, x_n) = 0 \quad \text{for} \quad i = 1, \ldots, m$$

$$\Updownarrow$$

$$\sum_{i=1}^{m} p_i^2(x_1, \ldots, x_n) = 0.$$

The complexity of
a) deciding whether there is a solution is roughly $d^{O(n)}$.

b) What does it even mean, to find a solution? One possibility is to use the *Thom encoding* of a real algebraic number: the minimal polynomial and signs of all its derivatives at the desired root. With that, the complexity is roughly $d^{O(n)}$.

c) The complexity of describing the set of solutions can be doubly exponential in n (computing Betti numbers). The problem can also be undecidable (homotopy type).

Systems of quadratic equations

If $d = 1$, we have a system of linear equations which can be solved in $O(n^3)$ time by Gaussian elimination.

What if $d = 2$? Quadratic equations are special.
First, any system of polynomial equations can be reduced to a system quadratic via substitutions of the type

$$y_{ij} := x_i x_j.$$

Second, some systems of quadratic equations naturally arise in applied problems.

Example (Distance Geometry, Computational Chemistry)

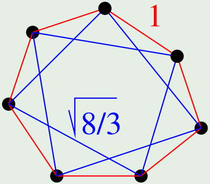
Question: Are there seven points $v_1, v_2, v_3, v_4, v_5, v_7$ in $\mathbb{R}^3$ such that

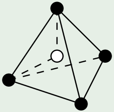
$$\|v_{(i+1) \mod 7} - v_i\| = 1 \quad \text{and} \quad \|v_{(i+2) \mod 7} - v_i\| = \sqrt{\frac{8}{3}}$$

for $i = 1, \ldots, 7$?

Systems of quadratic equations

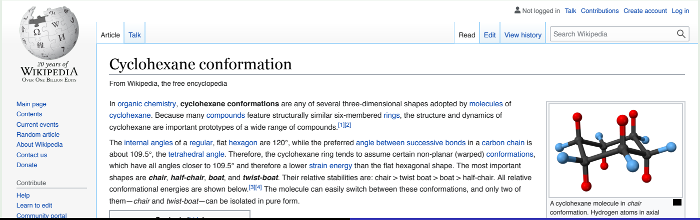
Example (Distance Geometry, Computational Chemistry)





The same question for six points.

Check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Cyclohexane conformation

From Wikipedia, the free encyclopedia

In organic chemistry, **cyclohexane conformations** are any of several three-dimensional shapes adopted by molecules of cyclohexane. Because many compounds feature structurally similar six-membered rings, the structure and dynamics of cyclohexane are important prototypes of a wide range of compounds.[1][2]

The internal angles of a regular, flat hexagon are 120°, while the preferred angle between successive bonds in a carbon chain is about 109.5°, the tetrahedral angle. Therefore, the cyclohexane ring tends to assume certain non-planar (warped) conformations, which have all angles closer to 109.5° and therefore a lower strain energy than the flat hexagonal shape. The most important shapes are ***chair***, ***half-chair***, ***boat***, and ***twist-boat***. Their relative stabilities are: chair > twist boat > boat > half-chair. All relative conformational energies are shown below.[3][4] The molecule can easily switch between these conformations, and only two of them—*chair* and *twist-boat*—can be isolated in pure form.

A cyclohexane molecule in *chair* conformation. Hydrogen atoms in axial

Systems of quadratic equations

Another example includes "trust region subproblems", see D. Bienstock, A note on polynomial solvability of the CDT problem, *SIAM J. Optim.* **26** (2016), no. 1, 488–498.

Results: A system of k quadratic equations in n real variables can be solved (questions a) and b) answered) in $n^{O(k)}$ time. In particular, if k is fixed in advance, in polynomial time.

Testing whether a system of homogeneous quadratic equations has a non-trivial solution: A. Barvinok, Feasibility testing for systems of real quadratic equations, *Discrete Comput. Geom.* **10** (1993), no. 1, 1–13.

Systems of quadratic equations

In the whole generality: D. Grigoriev and D.V. Pasechnik, Polynomial-time computing over quadratic maps. I. Sampling in real algebraic sets. *Comput. Complexity* **14** (2005), no. 1, 20–52.

For the description of the set of solutions (question c)), see S. Basu, D.V. Pasechnik, and M.-F. Roy, Bounding the Betti numbers and computing the Euler-Poincaré characteristic of semi-algebraic sets defined by partly quadratic systems of polynomials, *J. Eur. Math. Soc. (JEMS)* **12** (2010), no. 2, 529–553.

Positive semidefinite relaxation

We consider a system of quadratic equations

$$q_i(x) = a_i \quad \text{for} \quad i = 1, \ldots, m, \tag{1}$$

where $q_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ are quadratic forms,

$$q_i(x) = \langle Q_i x, x \rangle \quad \text{for} \quad i = 1, \ldots, m$$

for some $n \times n$ symmetric matrices $Q_1, \ldots, Q_m$.
In the space Sym_n of $n \times n$ symmetric matrices, we consider the inner product

$$\langle A, B \rangle = \mathrm{trace}(AB) = \sum_{i,j} a_{ij} b_{ij}$$

provided $A = (a_{ij})$ and $B = (b_{ij})$.
For $x \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, we define $x \otimes x$ as the $n \times n$ matrix $X = (x_{ij})$, where

$$x_{ij} = x_i x_j \quad \text{for all} \quad i, j.$$

Note that $X = x \otimes x$ if and only if X is positive semidefinite ($X \succeq 0$) and $\mathrm{rank}\, X \leq 1$.

Positive semidefinite relaxation

A vector $x \in \mathbb{R}^n$ is a solution to (1) if and only if $X = x \otimes x$ is a solution to

$$\langle Q_i, X\rangle = a_i \quad \text{for} \quad i = 1, \ldots, m.$$

Hence the system (1) is equivalent to the system

$$\begin{aligned} \langle Q_i, X\rangle &= a_i \quad \text{for} \quad i = 1, \ldots, m, \\ X &\succeq 0 \quad \text{and} \\ \operatorname{rank} X &\leq 1. \end{aligned} \tag{2}$$

If we remove the rank condition, the problem becomes convex: check whether the intersection of an affine subspace and the convex cone $\mathrm{Sym}_n^+ = \{X : \ X \succeq 0\}$ is non-empty.

Main question

Under some technical conditions (if the intersection is non-empty, it is not too small or too far away), the relaxed problem can be solved in polynomial time, see
Yu. Nesterov and A. Nemirovskii, *Interior-point Polynomial Algorithms in Convex Programming*, SIAM Studies in Applied Mathematics, **13**, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994.
Main question: When the existence of a solution to

$$\begin{aligned} \langle Q_i, X \rangle &= a_i \quad \text{for} \quad i = 1, \ldots, m \\ X &\succeq 0 \end{aligned} \tag{3}$$

guarantees the existence of a solution to (1).
This is the case, for example, when $m = 2$ or if $m = 3$, $n \geq 3$, and some linear combination of Q_1, Q_2, Q_3 is positive definite.

Dines Theorem

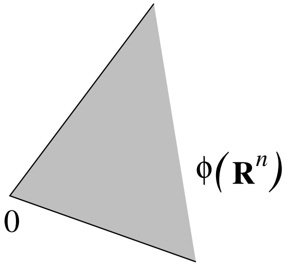
The case of $m = 2$ follows from the Dines Theorem,

Lloyd L. Dines, On the mapping of quadratic forms, *Bull. Amer. Math. Soc.* **47** (1941), 494–498.

The image of

$$\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^2, \qquad \phi(x) = \big(q_1(x), q_2(x)\big)$$

is a convex cone in $\mathbb{R}^2$.



Brickman Theorem

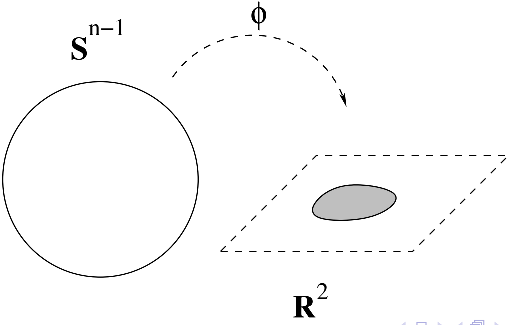
The case of $m = 3$ follows from the Brickman Theorem,

Louis Brickman, On the field of values of a matrix, *Proc. Amer. Math. Soc.* **12** (1961), 61–66.

if $n \geq 3$, $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ is a unit sphere, then the image of

$$\phi : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}^2, \qquad \phi(x) = \big(q_1(x), q_2(x)\big)$$

is a convex set in $\mathbb{R}^2$.



Fundamental counterexample

Generally, the existence of a solution to (3) does not imply the existence of a solution to (1).

Example

The system

$$x_1^2 = 1, \quad x_2^2 = 1, \quad 2x_1x_2 = 0$$

has no solutions, but the system

$$\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, X \right\rangle = 1, \quad \left\langle \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, X \right\rangle = 1, \quad \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, X \right\rangle = 0$$

has a solution $X \succeq 0$,

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Generally speaking

If $m < (r+2)(r+1)/2$, and (3) has a solution, there is a solution X of $\operatorname{rank} X \leq r$.
In addition, if $m = (r+2)(r+1)/2$ for $r > 0$, $n \geq r+2$ and the set of solutions to (3) is non-empty and compact, then there is a solution X with

$$\operatorname{rank} X \leq r.$$

See
A. Barvinok, Problems of distance geometry and convex properties of quadratic maps, *Discrete Comput. Geom.* **13** (1995), no. 2, 189–202
and
A. Barvinok, A remark on the rank of positive semidefinite matrices subject to affine constraints, *Discrete Comput. Geom.* **25** (2001), no. 1, 23–31.

Main goal

Our goal is to come up with a reasonably interesting (in particular, efficiently verifiable) sufficient condition when the existence of a solution to (3) implies the existence of a solution to (1).

$$\begin{matrix} \langle Q_i, X \rangle = \alpha_i \\ X \succeq 0 \end{matrix} \qquad \Longrightarrow \qquad q_i(x) = \alpha_i \quad \text{for} \quad i = 1, \ldots, m.$$

Change of variables

Let X be a solution of (3). Then $X = TT^*$ for some $n \times n$ matrix T, and so we have

$$a_i = \langle Q_i, X\rangle = \langle Q_i, TT^*\rangle = \operatorname{trace}(Q_i TT^*) = \operatorname{trace}(T^* Q_i T).$$

Let

$$\widehat{Q_i} = T^* Q_i T \quad \text{and} \quad \widehat{q}_i(x) = \langle \widehat{Q_i} x, x\rangle = q_i(Tx).$$

If x is a solution to

$$\widehat{q}_i(x) = a_i \quad \text{for} \quad i = 1, \ldots, m \tag{4}$$

then $y = Tx$ is a solution to (1). If X is invertible, then T is invertible and if y is a solution to (4) then $x = T^{-1}y$ is a solution to (1).

If *all* solutions X to (3) are not invertible, the system reduces to the case of an invertible X with fewer variables.

Restating the question

Note that $\operatorname{trace} \widehat{q}_i = a_i$. Hence ultimately we want to answer the following question:
Let $q_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ be quadratic forms,

$$q_i = \langle Q_i x, x \rangle \quad \text{for} \quad i = 1, \ldots, m.$$

When does the system

$$q_i(x) = \operatorname{trace} Q_i \quad \text{for} \quad i = 1, \ldots, m$$

have a solution $x \in \mathbb{R}^n$?

Enter Gaussian measure

Let us fix the standard Gaussian measure in $\mathbb{R}^n$ with density

$$\frac{1}{(2\pi)^{n/2}} e^{-\|x\|^2/2}.$$

Then for

$$q(x) = \langle Qx, x \rangle \quad \text{we have} \quad \mathbf{E}\, q(x) = \operatorname{trace}\, q.$$

Hence the equations

$$q_i(x) = \operatorname{trace}\, Q_i \quad \text{for} \quad i = 1, \ldots, m$$

are satisfied "on average".
Clearly, whether there is a solution, depends on the subspace $L = \operatorname{span}\{Q_1, \ldots, Q_i\}$, $L \subset \operatorname{Sym}_n$, rather than on the matrices $Q_1, \ldots, Q_m$ themselves.

An invariant of the subspace

Lemma

Let $L \subset \mathrm{Sym}_n$ *be a subspace and let* $A_1, \ldots, A_m$ *be an orthonormal basis of* L. *Then the matrix*

$$A(L) = A_1^2 + \ldots + A_m^2$$

depends only on L *and is independent on the choice of* $A_1, \ldots, A_m$.

An invariant of the subspace

Proof.

Let $B_1, \ldots, B_m$ be another orthonormal basis of L. Then

$$B_i = \sum_{j=1}^{m} \mu_{ij} A_j$$

for some orthogonal matrix (μ_{ij}). Now,

$$\sum_{i=1}^{m} B_i^2 = \sum_{i=1}^{m} \left(\sum_{j=1}^{m} \mu_{ij} A_j \right)^2 = \sum_{i=1}^{m} \left(\sum_{(j_1, j_2)} \mu_{ij_1} \mu_{ij_2} \right) A_{j_1} A_{j_2}$$

$$= \sum_{(j_1, j_2)} \left(\sum_{i=1}^{m} \mu_{ij_1} \mu_{ij_2} \right) A_{j_1} A_{j_2} = \sum_{j=1}^{m} A_j^2.$$

□

Main result

Theorem (1)

Let $q_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ be quadratic forms with matrices Q_i. Let $L = \operatorname{span}\{Q_1, \ldots, Q_m\}$, $L \subset \operatorname{Sym}_n$. If

$$\|A(L)\|_{\text{op}} \ \leq \ \frac{\gamma}{m},$$

then the system of equations

$$q_i(x) = \operatorname{trace} \, Q_i, \quad \textit{for} \quad i = 1, \ldots, m$$

has a solution $x \in \mathbb{R}^n$. Here $\| \cdot \|_{\text{op}}$ is the operator norm and $\gamma > 0$ is an absolute constant. One can choose $\gamma = 10^{-6}$.

The condition is satisfied, for example, when $m \leq \beta\sqrt{n}$ for some absolute constant $\beta > 0$ and random (for example, independent Gaussian) $Q_1, \ldots, Q_m$.

Main result

Theorem (1a)

Let $q_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ *be quadratic forms with matrices* Q_i *such that*

$$\operatorname{trace} Q_i = 0 \quad \text{for} \quad i = 1, \ldots, m.$$

Let $L = \operatorname{span}\{Q_1, \ldots, Q_m\}$, $L \subset \operatorname{Sym}_n$. *If*

$$\|A(L)\|_{\mathrm{op}} \ \leq \ \frac{\gamma}{m},$$

then the system of equations

$$q_i(x) = 0 \quad \text{for} \quad i = 1, \ldots, m$$

has a solution $x \in \mathbb{R}^n \setminus \{0\}$. *Here* $\| \cdot \|_{\mathrm{op}}$ *is the operator norm and* $\gamma > 0$ *is an absolute constant. One can choose* $\gamma = 10^{-6}$.

The idea of the proof of Theorem 1

Theorem (2)

Let $Q_1, \ldots, Q_m$ be $n \times n$ real symmetric matrices and let $\alpha_1 \ldots, \alpha_m$ be real numbers. Suppose that

$$\int_{\mathbb{R}^m} \det^{-\frac{1}{2}} \left(I - \mathbf{i} \sum_{i=1}^m \tau_i Q_i \right) \exp\left\{ -\mathbf{i} \sum_{i=1}^m \alpha_i \tau_i \right\} \, dt \neq 0,$$

where the integral converges absolutely.
Then the system of equations

$$\frac{1}{2}\langle Q_i x, x\rangle = \alpha_i \quad \text{for} \quad i = 1, \ldots, m$$

has a solution $x \in \mathbb{R}^n$.

Here $\mathbf{i}^2 = -1$, $t = (\tau_1, \ldots, \tau_m)$, and we pick a branch of $\det^{-\frac{1}{2}}\left(I - \mathbf{i} \sum_{i=1}^m \tau_i Q_i\right)$.

The idea of the proof of Theorem 1a

Theorem (2a)

Let $Q_1, \ldots, Q_m$ be $n \times n$ real symmetric matrices. Suppose that

$$\int_{\mathbb{R}^m} \det^{-\frac{1}{2}} \left(I - \mathbf{i} \sum_{i=1}^{m} \tau_i Q_i \right) \, dt \neq 0,$$

where the integral converges absolutely.
Then the system of equations

$$\langle Q_i x, x \rangle = 0 \quad \text{for} \quad i = 1, \ldots, m$$

has a solution $x \in \mathbb{R}^n \setminus \{0\}$.

Sketch of Proof of Theorems 2 and 2a

Let

$$q_i(x) = \frac{1}{2}\langle Q_i x, x\rangle \quad \text{for} \quad i = 1, \ldots, m.$$

We use that

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-q(x)}\, dx = \det^{-\frac{1}{2}} Q \quad \text{if} \quad Q \succ 0.$$

Therefore,

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp\left\{ \mathbf{i} \sum_{i=1}^{m} \tau_i q_i(x) \right\} e^{-\|x\|^2/2}\, dx = \det^{-\frac{1}{2}} \left(I - \mathbf{i} \sum_{i=1}^{m} \tau_i Q_i \right)$$

and

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp\left\{ \mathbf{i} \sum_{i=1}^{m} \tau_i \left(q_i(x) - \alpha_i \right) \right\} e^{-\|x\|^2/2}\, dx$$

$$= \det^{-\frac{1}{2}} \left(I - \mathbf{i} \sum_{i=1}^{m} \tau_i Q_i \right) \exp\left\{ -\mathbf{i} \sum_{i=1}^{m} \alpha_i \tau_i \right\}$$

Sketch of proof of Theorems 2 and 2a

Next, we use that

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\mathbf{i}a\tau} \exp\left\{-\frac{\tau^2}{2\sigma^2}\right\} \, d\tau = \exp\left\{-\frac{a^2\sigma^2}{2}\right\} \quad \text{for} \quad \sigma > 0.$$

Therefore,

$$(2\pi)^{\frac{n-m}{2}} \int_{\mathbb{R}^m} \det^{-\frac{1}{2}} \left(I - \mathbf{i} \sum_{i=1}^{m} \tau_i Q_i \right) \exp\left\{ -\mathbf{i} \sum_{i=1}^{m} \alpha_i \tau_i \right\} e^{-\|t\|^2/2\sigma^2} \, dt$$

$$= \sigma^m \int_{\mathbb{R}^n} \exp\left\{ -\frac{\sigma^2}{2} \sum_{i=1}^{m} (q_i(x) - \alpha_i)^2 \right\} e^{-\|x\|^2/2} \, dx.$$

Next, let

$$\sigma \longrightarrow +\infty.$$

□

The idea of the proof of Theorems 1 and 1a

Without loss of generality, we assume that $Q_1, \ldots, Q_m$ is an orthonormal basis of $L = \operatorname{span}\{Q_1, \ldots, Q_m\}$, $L \subset \operatorname{Sym}_n$. We consider the system of equations

$$q_i(x) = \alpha_i \quad \text{for} \quad i = 1, \ldots, m,$$

where

$$q_i(x) = \frac{1}{2}\langle Q_i x, x\rangle \quad \text{and} \quad \alpha_i = \frac{1}{2}\operatorname{trace} Q_i \quad \text{for} \quad i = 1, \ldots, m.$$

Let Q be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then for $\tau \approx 0$, we have

$$\det^{-\frac{1}{2}}(I - \mathbf{i}\tau Q) = \prod_{i=1}^{n}(1 - \mathbf{i}\tau\lambda_i)^{-\frac{1}{2}} = \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\ln(1 - \mathbf{i}\tau\lambda_i)\right\}$$

$$= \exp\left\{\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{\infty}\frac{(\mathbf{i}\tau\lambda_i)^k}{k}\right\} = \exp\left\{\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\operatorname{trace}(\mathbf{i}\tau Q)^k\right\}.$$

The idea of the proof of Theorems 1 and 1a

Consequently, for $t \approx 0$, $t = (\tau_1, \ldots, \tau_m)$, we have

$$\det^{-\frac{1}{2}}\left(I - \mathbf{i}\sum_{i=1}^m \tau_i Q_i\right) = \exp\left\{\frac{1}{2}\sum_{k=1}^\infty \frac{1}{k}\operatorname{trace}\left(\mathbf{i}\sum_{i=1}^m \tau_i Q_i\right)^k\right\}.$$

Now, since

$$\alpha_i = \frac{1}{2}\operatorname{trace}\, Q_i \quad \text{for} \quad i = 1, \ldots, m$$

then for $t \approx 0$, $t = (\tau_1, \ldots, \tau_m)$, we have

$$\det^{-\frac{1}{2}}\left(I - \mathbf{i}\sum_{i=1}^m \tau_i Q_i\right) \exp\left\{-\mathbf{i}\sum_{i=1}^m \alpha_i \tau_i\right\}$$

$$\approx \exp\left\{-\frac{1}{4}\operatorname{trace}\left(\sum_{i=1}^m \tau_i Q_i\right)^2\right\} = \exp\left\{-\frac{1}{4}\sum_{i=1}^m \tau_i^2\right\},$$

and we show that the contribution of a neighborhood of $t = 0$ dominates the integral.

Some related integrals

Here is an idea how to argue that systems of k homogeneous quadratic equations are simple, provided k is fixed in advance. This is *not* how it has been done, but it shows a useful underlying algebraic structure.

Let

$$q_i(x) = \langle x, Q_i x \rangle \quad \text{for} \quad i = 1, \ldots, k,$$

where Q_i are $n \times n$ symmetric matrices and $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$.

Let

$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \quad \|x\| = 1\}$$

be the unit sphere endowed with the rotation invariant Borel probability measure μ.

The generating function

Theorem

In a neighborhood of $z_1 = \ldots = z_k = 0$, *we have*

$$\det^{-\frac{1}{2}}\left(I - \sum_{i=1}^{k} z_i Q_i\right) = \sum_{m_1,\ldots,m_k \geq 0} a_{m_1,\ldots,m_k} z_1^{m_1} \cdots z_k^{m_k},$$

where

$$a_{m_1,\ldots,m_k} = \frac{\Gamma\left(m_1 + \ldots + m_k + \frac{n}{2}\right)}{m_1! \cdots m_k! \Gamma\left(\frac{n}{2}\right)} \times \int_{\mathbb{S}^{n-1}} q_1^{m_1}(x) \cdots q_k^{m_k}(x) \; d\mu(x).$$

Proof: We note that for

$$q(x) = \langle x, Q_x \rangle,$$

in a neighborhood of $z = 0$, we have

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{zq(x)/2} e^{-\|x\|^2/2} \, dx = \det^{-\frac{1}{2}} (I - zQ) \, .$$

Consequently,

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{(z_1 q_1(x) + \ldots + z_k q_k(x))/2} e^{-\|x\|^2/2} \, dx = \det^{-\frac{1}{2}} \left(I - \sum_{i=1}^{k} z_i Q_i \right) .$$

Expanding into the Taylor series in a neighborhood of $z_1 = \ldots = z_k = 0$, we get

$$\det^{-\frac{1}{2}} \left(I - \sum_{i=1}^{k} z_i Q_i \right) = \sum_{m_1, \ldots, m_k \geq 0} b_{m_1, \ldots, m_k} z_1^{m_1} \cdots z_k^{m_k},$$

where

$$b_{m_1,\ldots,m_k} = \frac{1}{2^{m_1+\ldots+m_k} m_1! \cdots m_k!} \times \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} q_1^{m_1}(x) \cdots q_k^{m_k}(x) e^{-\|x\|^2/2} \, dx.$$

For a homogeneous polynomial $F(x)$ of degree $2m = 2m_1 + \ldots + 2m_k$, we have

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} F(x) e^{-\|x\|^2/2} \, dx = \frac{2^m \Gamma\left(m + \frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \int_{\mathbb{S}^{n-1}} F(x) e^{-\|x\|^2/2} \, d\mu(x).$$

□

The generating function

Corollary: The integral

$$\int_{\mathbb{S}^{n-1}} q_1^{m_1}(x) \cdots q_k^{m_k}(x)\ d\mu(x)$$

can be computed in $n^{O(1)} \left(m_1 + \ldots + m_k\right)^{O(k)}$ time.

If $q_1, \ldots, q_k : \mathbb{R} \longrightarrow \mathbb{R}$ are positive semidefinite, then for large m,

$$\left(\int_{\mathbb{S}^{n-1}} \left(q_1(x) \ldots q_k(x)\right)^m\ d\mu(x)\right)^{1/m} \approx \max_{x \in \mathbb{S}^{n-1}} q_1(x) \cdots q_k(x).$$

In fact, to approximate the maximum within relative error ϵ, we can choose $m = O\left(\frac{n+km}{\epsilon}\right)$.

Remark: If k is fixed in advance, we can do it in polynomial time *exactly*.

The generating function

Connection to feasibility: Given quadratic forms $q_1, \ldots, q_k : \mathbb{R}^n \longrightarrow \mathbb{R}$, let us define

$$q_i^+ = \|x\|^2 + \epsilon q_i(x) \quad \text{and} \quad q_i^- = \|x\|^2 - \epsilon q_i(x) \quad \text{for} \quad i = 1, \ldots, k$$

and some small $\epsilon > 0$.
Then

$$\begin{aligned} &\max_{x \in \mathbb{S}^{n-1}} q_1^+(x) \cdots q_k^+(x) q_1^-(x) \cdots q_k^-(x) \\ &= \max_{x \in \mathbb{S}^{n-1}} \left(1 - \epsilon^2 q_1^2(x)\right) \cdots \left(1 - \epsilon^2 q_k^2(x)\right) \\ &= \begin{cases} 1 & \text{if} \quad q_i(x) = 0 \quad \text{for some } x \in \mathbb{S}^{n-1} \quad \text{and all } i \\ < 1 & \text{otherwise.} \end{cases} \end{aligned}$$

Computing the integral may allow us to estimate the volume of the set of solutions.