# Tropicalization in Convex Geometry and Combinatorics

Joint with

Annie Raymond, Mohit Singh, Rekha Thomas

and

Felipe Rincon, Rainer Sinn, Cynthia Vinzant, Josephine Yu.

$S$ - "complicated" set $\quad S \subseteq \mathbb{R}^n$

Hadamard property

$$x = (x_1, \dots, x_n) \qquad x \circ y = (x_1 y_1, \dots, x_n y_n)$$

$$y = (y_1, \dots, y_n)$$

$$x, y \in S \Rightarrow x \circ y \in S$$

"Partial Information"

Binomial inequalities valid on $S$.

$$x^\alpha \geq x^\beta \qquad x^\alpha - x^\beta \geq 0$$

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} \geq x_1^{\beta_1} \cdots x_n^{\beta_n}$$

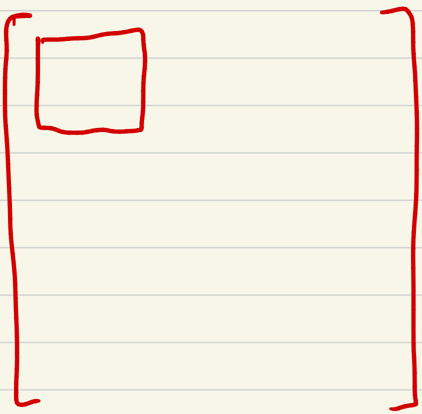
<u>Example:</u> $S$ = cone of PSD matrices

$$A \in \mathrm{Sym}^{n \times n}.$$

$x^T A x \geq 0$ for all $x$ in $\mathbb{R}^n$

Schur Product Theorem:

$S^n_+$ has Hadamard property.



$$\begin{matrix} d_1 & a \\ a & d_2 \end{matrix}$$

$$d_1 d_2 - a^2 \geq 0$$

These inequalities generate all binomial inequalities valid on $S^n_+$.

$x^\alpha \geq x^\beta \qquad S \subseteq \mathbb{R}^n_{\geq 0}$

$\langle \alpha, \log x \rangle \geq \langle \beta, \log x \rangle$

$\langle \alpha - \beta, \log x \rangle \geq 0$

$S \xrightarrow{\log} \log(S)$ linear inequalities valid on $\log S$

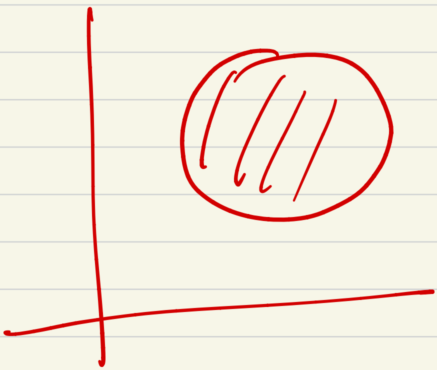
$\underbrace{\operatorname{cone}(\log(S))^*}$ ← linear inequalities valid on $\log(S)$

If $S$ has Hadamard property, then

$$\underbrace{\overline{\operatorname{cone}(\log(S))} = \operatorname{trop}(S)}$$

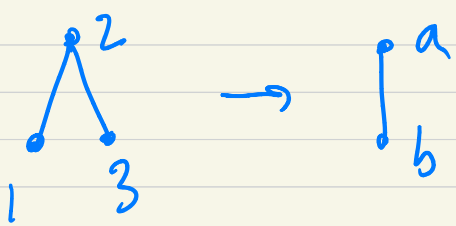
$$\operatorname{trop}(S) = \lim_{t\to\infty} \log_t(S).$$

If $S$ is semialgebraic $\Rightarrow$ $\operatorname{trop}(S)$ is a rational polyhedral complex.



# Combinatorics:

$\varphi : H \to G$ is a homomorphism
if $\varphi$ sends edges to edges.




$2 \to a$

$1, 3 \to b$

$1, 2 \to a$

$3 \to b$ X

$H_1, \dots, H_k$ build set $S$ in $\mathbb{R}^k$

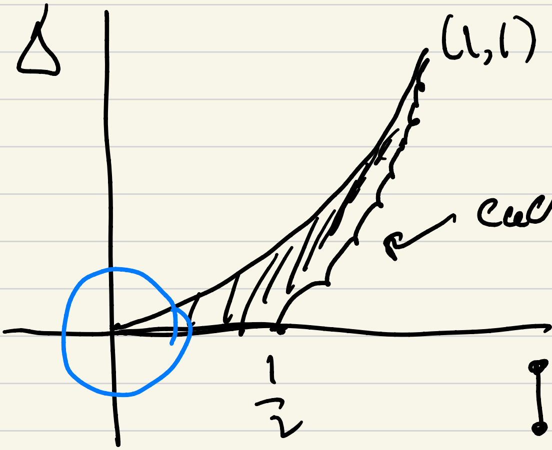
for all graphs $G$ build a vector

$$h_G = (\#\mathrm{Hom}(H_1, G), \dots, \#\mathrm{Hom}(H_k, G))$$

$$S = \{h_G : G \text{ graph}\}$$

$$\tilde{h}_G = \left\{ \frac{\#\mathrm{Hom}(H_i, G)}{|H_i|^{|G|}} \right)$$

$S \subseteq [0,1]^k$ Razborov




$\delta^3 \geq \Delta^2$

each scallop is a piece of a cubic polynomial

infinitely many.

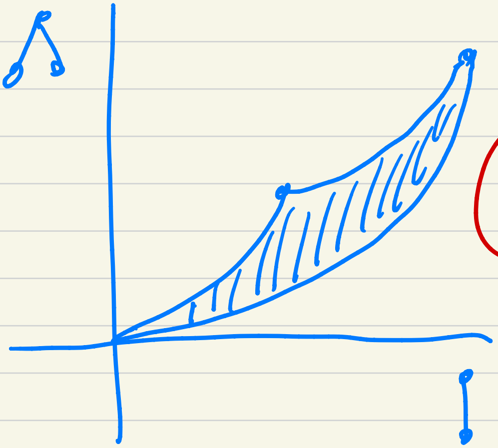
S has Hadamard property

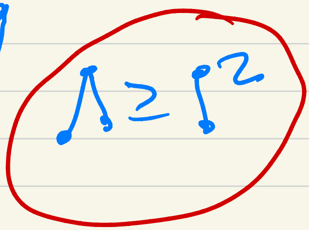
since $\#\mathrm{hom}(H, G_1) \cdot \#\mathrm{hom}(H, G_2)$

$\shortparallel$

$\#\mathrm{hom}(H, G_1 \otimes G_2)$

$A_{G_1} \otimes A_{G_2}$





 $, \triangle, K_4, K_5, \ldots, K_n$

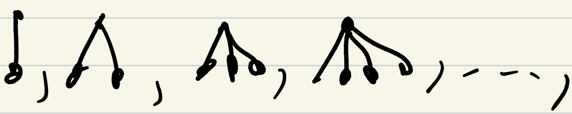
$S$ - low-density profile

$K_a^b \geq K_b^a$

$a \leq b$

 $\leq 1$

Kruskal–Katona inequalities



## Sidorenko Conjecture:

$H$ - bipartite graph $\quad H \geq I^{|E(H)|}$

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Operations that preserve Hadamard property:

$S$ - has Hadamard property

(1) $\varphi_{\mathcal{A}}(S) \qquad \mathcal{A} = \{ v \in \mathbb{Z}^n \}$

$$x \longmapsto \left( x^{v_1}, \dots, x^{v_k} \right) \qquad v_1, \dots, v_k$$

(2) $\operatorname{conv}(S)$ has Hadamard property

(3) $L \cap S \quad L$ - linear subspace

of the form $(x, \dots, x, y, \dots, y, z, \dots z, \dots)$

Schur product theorem: $S = \mathbb{R}^n$

$\mathcal{A} = \{ \theta \in \mathbb{R}^n, \ \theta_i \geq 0, \ \sum \theta_i = 2 \}$

$\varphi_{\mathcal{A}}(S) =$ rank 1 matrices

$$x \to \begin{matrix} x_1^2 & & x_i x_j \\ & \ddots & \\ x_i x_j & & x_n^2 \end{matrix}$$

$\mathrm{conv}(\varphi_{\mathcal{A}}(S)) =$ cone of PSD matrices.

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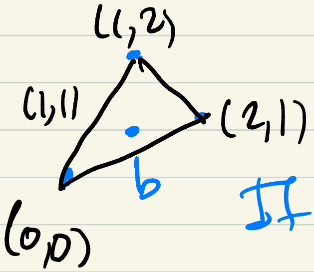
What happens if we take a different set of monomials $\mathcal{A}$?

$$S = \mathbb{R}^n_{\geq 0} \to \varphi_{\mathcal{A}}(S) \to \underbrace{\mathrm{conv}(\varphi_{\mathcal{A}}(S))}_{\substack{\| \\ \mathcal{M}_{\mathcal{A}}}}$$

**Thm:** trop$(\mathcal{M}_{\mathcal{A}})$ = functions on $\mathcal{A}$ that are convex.

$f$ is extendable to a globally convex functions




If $b = \sum \lambda_i a_i$ $\quad b, a_i \in \mathcal{A}$

then $f(b) \le \sum \lambda_i f(a_i)$

Valid binomial inequalities on $\mathcal{M}_{\mathcal{A}}$ come from points in $\mathcal{A}$ in non-convex position

$(\lambda_i, \dots, -1)$

$$m_{00} \cdot m_{12} \cdot m_{21} \geq m_{11}^3$$

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What is $\mathcal{M}_A$?

$\shortparallel$

The cone of $A$-moments of measures supported on $\mathbb{R}^n_{\geq 0}$.

$v \in A$ $\quad \displaystyle\int_{\mathbb{R}^n_{\geq 0}} x^v \, d\mu$ $\quad \mu$ - measure

$p = (p_1, \dots, p_n) \in \mathbb{R}^n$

$\varphi_A(p) = (p^{v_1}, \dots, p^{v_k})$ records moments of a measure

$\delta$ - measure centered at $p$.

$\mathrm{Cone}(\varphi_{\mathcal{A}}(S))$ = moments of finitely supported measures of $S$.

But any measure can be "faked" by a finitely supported measure...

Truncated Moment problem.

Extreme Rays of $\mathcal{M}_{\mathcal{A}}(S)$ are point evaluations on $S$.

$\mathcal{M}_{\mathcal{A}}(S)^* \equiv$ polynomials supported on $\mathcal{A}$.

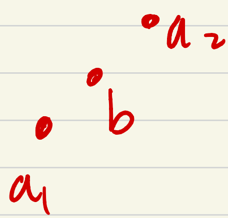
Nonnegative polynomials on $\mathcal{A}$ supported on $S$.

$\sum x_i \cdot SOS$ ← Sums of squares on $\mathbb{R}^n_{\geq 0}$

$\Sigma^* =$ pseudo-moments

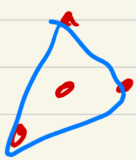
$\Sigma^*$ – also has Hadamard property

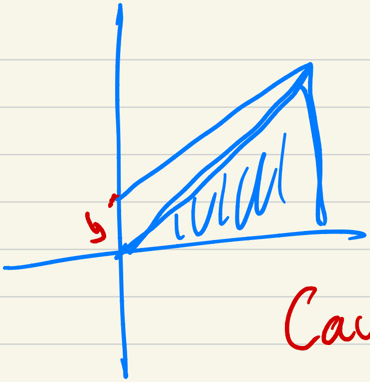
$\Sigma^* =$ mid-point convex inequalities.




$$\frac{a_1+a_2}{2} = b$$

$$\frac{1}{2} f(a_1) + \frac{1}{2} f(a_2) \geq f(b)$$





$$\Delta^2 \geq \int^3$$

Cauchy - Schwartz

Calculus

$$\Pi - \int^3 \geq 0$$