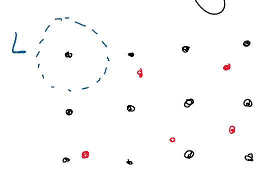
On the complexity of unconditional convex bodies

Thursday, April 15, 2021

Barvinok and Veomett question: Can any n-dimensional convex body be approximated by the projection of a section of an N-dimensional simplex with controlled number of facets?

- Rekha Thomas (on the previous week)
  - → Lifts
  - → Cross-Polytope: $\begin{cases} C_n = \{x \in \mathbb{R}^n : \sum \pm x_i \leq 1\} \to 2^n \text{ facets} \\ Q_n = \{(x,y) \in \mathbb{R}^{2n} : \sum y_i = 1, -y_i \leq x_i \leq y_i \ \forall i\} \Rightarrow C_n = \Pi_x(Q_n) \text{ for } \Pi_x(x,y) = x. \end{cases}$
    - ↳ $2n$ facets.

Answer: Not in general



- Litvak + Rudelson + Tomczak-Jaegermann '14: $\exists L \in \mathcal{K}^n, L = -L$, s.t. $\forall K \in \mathcal{K}^n$ produced as a projection of a section of S in $\mathbb{R}^N$

$$d(L,K) \geq c \sqrt{\frac{n}{\log \frac{2N \log 2N}{n}}}, \quad c \text{ abs. constant}$$

→ To prove this they estimated the complexity of the set of all Banach spaces

Pisier '13: Such complexity is doubly exponential in n.

- See also survey by Mankiewicz + Tomczak-Jaegermann (Quotients of finite-dimensional Banach spaces)

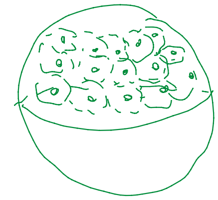
- Rudelson '16:
  - → Unconditional
  - → Completely symmetric

Def: Let (X,d) be a metric space, $\varepsilon>0$, then

- $N_\varepsilon \subset X$ is an ε-net if $\forall x\in X, \exists y\in N_\varepsilon : d(x,y)\le\varepsilon$.
- If X is compact then the minimal cardinality, $N(X,\varepsilon)$, of a net is called the covering number
- $\log_2 N(X,\varepsilon)$ is the metric entropy of X In geometric functional analysis

In some sense it measures the complexity of the metric space.

Nets are a convenient way to discretize compact sets



E.g (Covering number of S^{n-1}):

Let $\varepsilon>0$, and N_ε a maximal ε-separated set ($\forall x,y\in N_\varepsilon$, $d(x,y)>\varepsilon$ and no other subset of N_ε has this property)

- N_ε is a net: If $\exists x\in X$ s.t. $\forall y\in N_\varepsilon$ $d(x,y)>\varepsilon \Rightarrow N_\varepsilon\cup\{x\}$ is ε-separated.
- $B_2^n(x,\frac{\varepsilon}{2})\cap B_2^n(y,\frac{\varepsilon}{2})=\phi$ $\forall x,y\in N_\varepsilon, x\ne y$.
- $B_2^n(x,\frac{\varepsilon}{2})\subset(1-\frac{\varepsilon}{2})B_2^n$

$\Rightarrow$ $(\frac{\varepsilon}{2})^n|B_2^n|\cdot|N_\varepsilon| = |\frac{\varepsilon}{2}B_2^n|\cdot|N_\varepsilon| \le |(1+\frac{\varepsilon}{2})B_2^n| = (1+\frac{\varepsilon}{2})^n|B_2^n|$

$\Rightarrow |N_\varepsilon|\le(1+\frac{2}{\varepsilon})^n$

E.g. (Spectral norm on a net):

Let A be an $N\times n$ matrix, N_ε a net for S^{n-1}, $\varepsilon\in(0,1)$. Then,

- $\|A\| = \sup_{x\in S^{n-1}}\{\|Ax\|_2\} \ge \max_{x\in N_\varepsilon}\{\|Ax\|_2\}$
- Let $y\in S^{n-1}$: $\|A\| = \|Ay\|_2 \Rightarrow \exists x\in N_\varepsilon : \|x-y\|_2\le\varepsilon$

  $\Rightarrow \|Ax-Ay\|_2 \le \|A\|\cdot\|x-y\|_2 \le \varepsilon\cdot\|A\|$.

Thus,

$\|A\| \ge \|A\| \ \ \|Ax-Ay\|_2 \ge \|A\|-\varepsilon\|A\| = (1-\varepsilon)\|A\|$

Thus,

$$\|Ax\|_2 \geq \|Ay\|_2 - \|Ax - Ay\|_2 \geq \|A\| - \varepsilon\|A\| = (1-\varepsilon)\|A\|.$$

Then,

$$\max_{x \in N_\varepsilon} \{\|Ax\|_2\} \leq \|A\| \leq (1-\varepsilon)^{-1} \max_{x \in N_\varepsilon} \{\|Ax\|_2\}$$

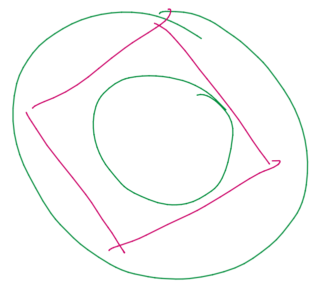
Thus, instead of taking supremum over the whole sphere to compute the spectral norm, we can estimate it on a net.

Def: Let X, Y be two n-dim normed spaces,

Geometric

$$d_{BM}(X,Y) = \inf\{\frac{R}{r} : r\underbrace{B_Y \subseteq T B_X}_{\text{unit balls of } Y \text{ and } X} \subseteq R B_Y, \text{ for some } T \in GL(n)\}$$

Correspondence between normed spaces and origin symm. convex bodies



Analytic

$$d_{BM}(X,Y) = \inf\{\|T\| \cdot \|T^{-1}\| : T : X \to Y \text{ isomorphism}\}$$

The BM distance defines a metric on a set of equivalent classes of normed spaces. It shows how far the spaces are from being isometric.

Remarks:

1. $d(X,Y) \geq 1$
2. $d(X,Y) = d(Y,X)$
3. $d(X,Z) \leq d(X,Y)\, d(Y,Z)$

} Resembles a metric

4. $\log d_{BM}$ defines a metric on $\mathcal{B}_n$ ← set of all n-dim Banach spaces.
5. $(\mathcal{B}_n, d_{BM})$ is a compact metric space ← Banach Mazur Compactum

John '48: $d(X, \ell_2^n) \leq \sqrt{n}$, $\forall X \in \mathcal{B}_n$. In particular,

$d(\ell_1^n, \ell_2^n)$

$$\sqrt{n} \overset{?}{\leq} d(X,Y) \leq n \quad \forall X,Y \in \mathcal{B}_n$$

↳ By multiplicative triangle ineq.

<u>Gluskin '81</u>: $\exists\, c>0$, abs. constant, $\forall n \in \mathbb{N}$, $\exists\, X,Y \in \mathcal{B}_n$ with $d(X,Y) \geq c\cdot n$.

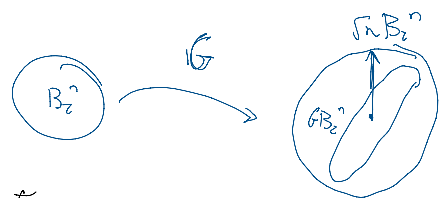
<u>Remark</u>: diameter of $\mathcal{B}_n \simeq n$.

Gluskin's Idea:

Let $G = [g_1 \cdots g_n]$ Gaussian matrix, $g_i \sim N(0, I_n)$ $\forall i \in [n]$, $g_i = (\gamma_{i1}, \dots, \gamma_{in})$ for $\gamma_{ij} \sim N(0,1)$.

smallest scaling needed of the ball

- $\mathbb{E}\| G : \ell_2^n \to \ell_2^n \| = \mathbb{E} \sup_{x \in S^{n-1}} \|Gx\|_2 \simeq \sqrt{n}$



☒ For the lower bound we plug a vector, and $\mathbb{E}\|g_i\|_2 \simeq \sqrt{n}$

☒ For the upper bound, we discretize on the sphere as on the previous example and we use Gaussians decay very fast (concentration).

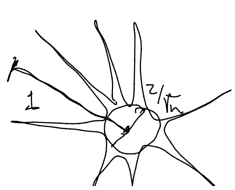
- $\mathbb{E}\| G : \ell_1^n \to \ell_1^n \| = \mathbb{E} \sup_{i \leq n} \{\|Ge_i\|_1\} \geq n \simeq n$

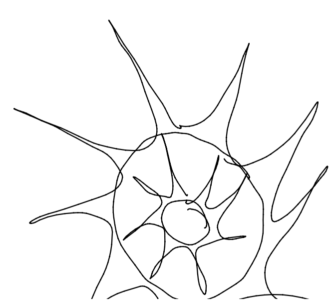
by convexity is on a vertex

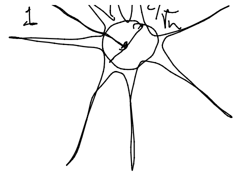
$n\sqrt{\frac{2}{\pi}}$

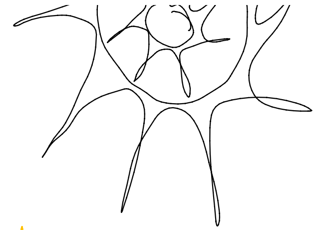
$$\boxed{\text{Idea: Put the randomness in the space}}$$

what if we throw some extra points to ℓ_1^n?









- $|B_1^n|^{1/n} \simeq \frac{c}{n}$
- $|B^n_{1/\sqrt{n}}|^{1/n} = \left[\left(\frac{1}{\sqrt{n}}\right)^n |B_1^n|\right]^{1/n} \simeq \frac{1}{\sqrt{n}} \cdot \frac{c}{\sqrt{n}} \simeq \frac{c}{n}$

The feature of these bodies is that they have a bulk that makes up most of the volume, with little diameter, and spikes with little volume but big diameter.

Gluskin shows: $\forall n \in \mathbb{N}$, $\exists X_n, Y_n \subset \mathcal{B}_n$ s.t. $\forall T$ isomorphism

$$\left.\begin{array}{l} \|T: X_n \to Y_n\| \geq c\sqrt{n} \\ \|T: Y_n \to X_n\| \geq c\sqrt{n} \end{array}\right\} \Rightarrow d_{BM}(X_n, Y_n) \geq c \cdot n$$

Outline of the construction:

$$B_{X_n} = \operatorname{Conv}\{\underbrace{\pm \frac{g_1}{\sqrt{n}}, \ldots, \pm \frac{g_{2n}}{\sqrt{n}}}_{\text{random points}}, \underbrace{\pm e_1, \ldots, \pm e_n}_{\text{Deterministic}}\} \text{ for } g_1, \ldots, g_{2n} \sim N(0, I_n)$$

?

$$B_{Y_n} = \operatorname{Conv}\{\pm \frac{g_1'}{\sqrt{n}}, \ldots, \pm \frac{g_{2n}'}{\sqrt{n}}, \pm e_1, \ldots, \pm e_n\}$$

- We divide by $\sqrt{n}$ so we're roughly on the sphere, just normalization.
- If we didn't add $\pm e_i$'s we'd need the inradius of the ball of all Gaussian vectors. Including the ℓ_1^n ball automatically gives us a ball of radius $\frac{1}{\sqrt{n}}$.
- In particular, as we need to deal with all operators from one space to the other, we want to be able to consider one net for all of them.

The identity:

by independence

$$\mathbb{P}(B_{X_n} \subseteq RB_{Y_n}) \le \mathbb{P}\left(\pm \frac{g_i}{\sqrt{n}} \in RB_{Y_n}\ \forall i \in [2n]\right) \le \prod_{i=1}^{2n} \mathbb{P}(\pm g_i \in \sqrt{n} R B_{Y_n})$$

by independence

Convex set so include endpoints

$$= [\mathbb{P}(g_1 \in \sqrt{n} R B_{Y_n})]^{2n}$$

identically distr.

$$\mathbb{P}(g_1 \in \sqrt{n} R B_{Y_n}) \le \frac{1}{(2\pi)^{n/2}} \int_{\sqrt{n} R B_{Y_n}} e^{-\frac{\|x\|^2}{2}} dx \le c^n |\sqrt{n} R B_{Y_n}| \le c^n \frac{(\sqrt{n}R)^n}{n^n}$$

c/n ?

integrate on the set

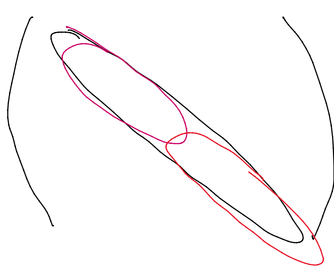
bound density by 1

even when R is of the order of $\sqrt{n}$ the probability is small.

Intuitively $|B_{Y_n}| \sim \frac{c}{n}$, the set without the Gaussians is the ℓ_1^n-ball which has small volume, if we add the Gaussians, and they are on the sphere, then adding finitely many of them is going to have small volume.

Say now $T \neq I \Rightarrow$ SVD so $T = UDV^t$

we're going to assume half of the diagonal values are big and half are small, since when considering T^{-1} we take the reciprocals and the same follows for T^{-1}



In general if T is very degenerate it'd be hard to have a good small ball, but on the inverse we'd have a very good one. So then we just order them and scale the the middle ones are roughly equal.

$$TB_{X_n} \subseteq RB_{Y_n} \Rightarrow P_E B_{X_n} \subseteq P_E B_{Y_n} \Rightarrow P_E T g_i \in \sqrt{n} P_E R B_{Y_n} \Rightarrow T P_F g_i \in \sqrt{n} P_E B_{Y_n}$$

$\mathrm{span}\{u_1, \dots, u_{n/2}\}$

$\mathrm{span}\{v_1, \dots, v_{n/2}\}$

inside the other holds for proj

Suppose T is diagonal

$$\Rightarrow P_F g_i \in \sqrt{n} R \tilde{T}^{-1} P_E B_{Y_n} \Rightarrow P_F g_i \in \sqrt{n} R P_E B_{Y_n}$$

Projection of a gaussian is a gaussian

$\tilde{T} = T|_F$

Thus,

2n

Thus,

- $\mathbb{P}(TB_{X_n} \subseteq RB_{Y_n}) \le \prod_{i=1}^{2n} \mathbb{P}(P_F g_i \in \sqrt{n} R P_E B_{Y_n}) = |\sqrt{\pi} P_E B_{Y_n}|^{2n} \le (c^{n/2})^{2n} = c^{n^2}$

  ↑ $\frac{n}{2}$-dim volume

- $|N_1| \le C^{n^2}$

$\mathbb{P}(\{T : TB_{X_n} \subseteq RB_{Y_n}\}) \le c_1^{n^2} \cdot C^{n^2} \approx c_2^{n^2}$ since we cook up the constants to make some decay

---

Main Result: Let $1 \le \varepsilon \le \tilde{c}\, n^{1/2} \log^{-5/2} n$, $\mathcal{K}_n^{unc}$ contains a ε-separated set of cardinality at least

$$\exp\left(\exp\left(\frac{c}{\varepsilon^2 \log^4(\varepsilon^2+1)}\, n\right)\right)$$

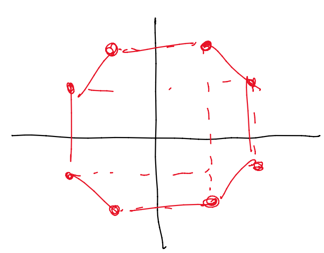
Corollary: Let $n \in \mathbb{N}$, $\exists L \in \mathcal{K}_n^{unc}$ s.t. $\forall K \in \mathcal{K}^n$ projection of section of simplex $\subset \mathbb{R}^N$ then

$$d(L,K) \ge \underset{\substack{\uparrow \\ \text{abs. constant}}}{c} \left(\frac{n}{\log N}\right)^{1/4} \cdot \log^{-1}\left(\frac{n}{\log N}\right)$$

Def: Let $x_1, \dots, x_m \in \mathbb{R}^n$ with $x_j = (x_{j1}, \dots, x_{jn})$ then

$$\text{unc.conv}\{x_1, \dots, x_m\} = \text{conv}\{(\varepsilon_{11} x_{11}, \dots, \varepsilon_{1n} x_{1n}), \dots, (\varepsilon_{m1} x_{m1}, \dots, \varepsilon_{mn} x_{mn})\}$$

where $\varepsilon_{ij} = \pm 1$.



Rudelson's

$$K = K(\underbrace{I_1, \dots, I_N}_{\text{random sets}})$$

$$= \text{abs.conv}\{\text{unc.conv}\{x_1, \dots, x_N\}, \sqrt{\delta n} B_1^n, \delta\sqrt{n} B_2^n\}$$

$= \text{abs.} \operatorname{Conv}\{ \operatorname{abs.}\operatorname{Conv}\{x_1, \dots, x_N\}, \underbrace{\sqrt{\delta n}\, B_1^n, \sqrt{\delta n}\, B_2^n}_{\text{deterministic}} \}$

I_ℓ's ind. random sets in $[n]$

unif. chosen among sets with cardinality δn, $\delta > 0$.

$x_\ell = \sum_{j \in I_\ell} e_j \longleftarrow$ Coordinates are not independent

$$\boxed{\begin{aligned} &\mathbb{P}\Big(\sum_{j \in J \subset [n]} \varepsilon_j e_j \in R_\delta' K'\Big) \le \exp(-c' \delta^2 n) \longleftarrow 4.1 \\ &\hookrightarrow \mathbb{P}(TK \subseteq R_\delta K') \le \exp(-\exp(c \delta' n)) \longleftarrow 4.2 \end{aligned}}$$

ε_i's symmetric ± 1 Bernoulli random variables $\sum \to$ Decays as Gaussian

J random subset $\subset [n]$ with cardinality δn.

<u>Intuition:</u> $\mathbb{P}(Y \in 2\operatorname{Conv}\{B_2^{I_\ell}\}_{i=1}^N)$

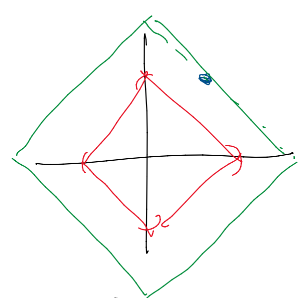
Y is δ's $\quad B_2^{I_\ell}$ is δ's Coordinates missalign so we need many (Glisten spikes)

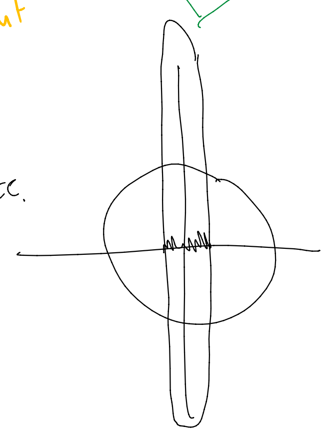
$\mathbb{P}(g \in \mathcal{E}) = \mathbb{P}(g \in T B_2^n) = \mathbb{P}(T^{-1} g \in B_2^n)$

but $T^{-1} g = \langle T^{-1/2} g, T g^{-1/2} \rangle$ $\quad T$ diagonal $\quad$ Hanson-wright ineq.

<u>Key idea:</u> we must consider the geometry of target space.

$\Rightarrow$ we only treated the I $\quad T$ $\quad$ N blocks





$\mathbb{P}(TK \subseteq R_\delta K') \le \prod^{N} \mathbb{P}(T I_\ell \dots) \le [\dots]^N$

$$\mathbb{P}(TK \subset R_\delta K') \leq \prod_{i=1}^{N} \mathbb{P}(T\sum \varepsilon_j e_j \in R_\delta K') = [\exp(-c'\delta^2 n)]^N$$

$$= \exp(-N c'\delta^2 n) = \exp(-c'\delta^2 n \exp(c\delta^2 n))$$

$N = e^{c\delta^2 n}$

$$\leq \exp(-\exp(c\delta^2 n))$$

$|N| \leq C^{n^2}$

$$\mathbb{P}(\exists T : TK \subseteq R_\delta K') \leq |N_{\frac{\varepsilon}{r\delta}}| \exp(-\exp(c'\delta^2 n))$$

$$\leq \exp(-\exp(c'\delta^2 n))$$

Thus,

$$\exp\left(\exp\left(\frac{cn}{(2d)^4 \log((2d)^2-1)}\right)\right) \leq \# \{\text{set of uncond. as a projection of a section}\} \leq 3\log N$$

Rudelson's Theorem

LRTJ's result.