Polynomial Time Algorithms in Invariant Theory for Torus Actions

M. Levent Doğan

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Based on a joint work with P. Bürgisser, V. Makam, M. Walter, A. Wigderson

Permanent vs Determinant

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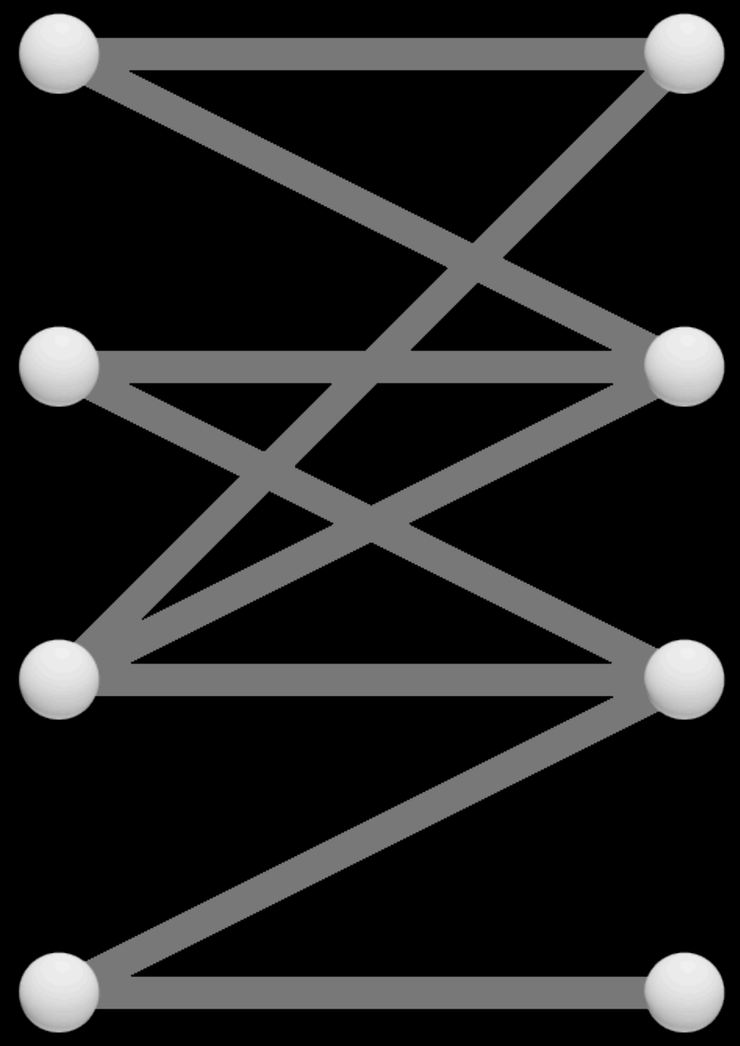
- Gaussian elimination

$$\text{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

G



Biadjacency matrix →

$\text{Adj}(G)$

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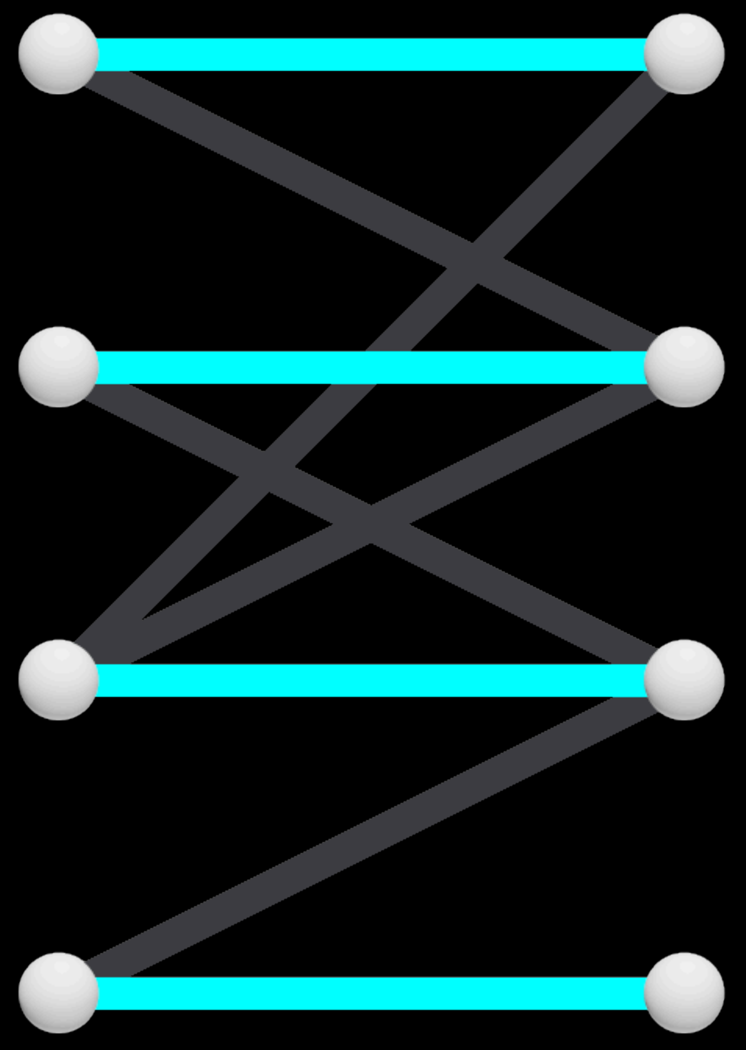

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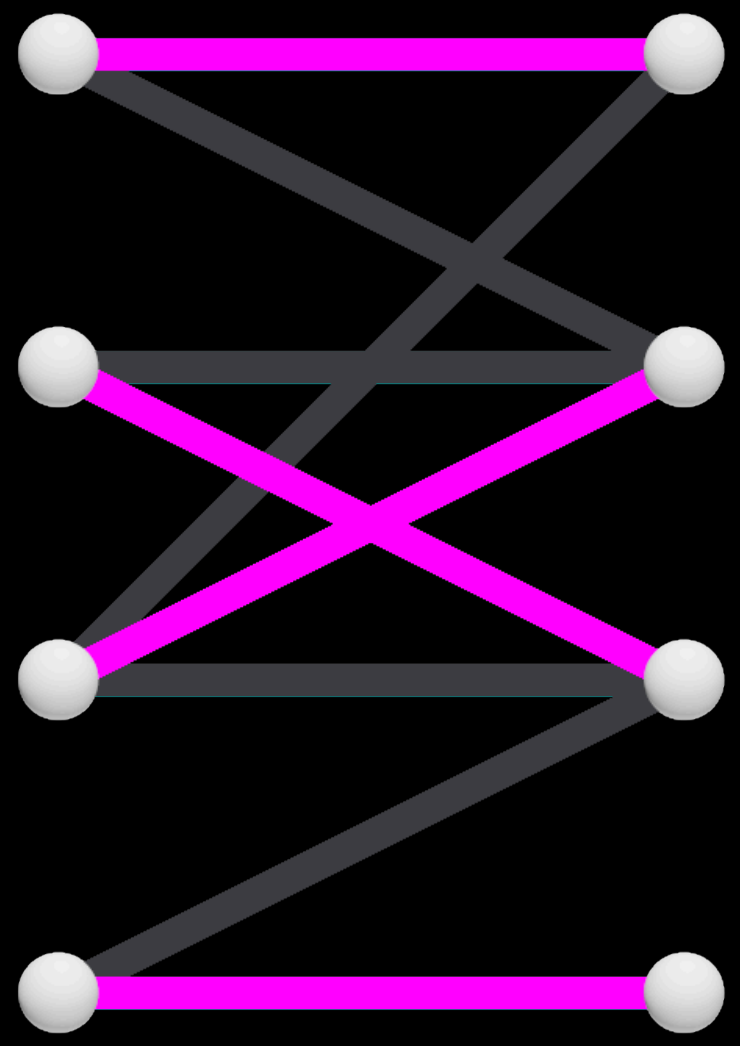

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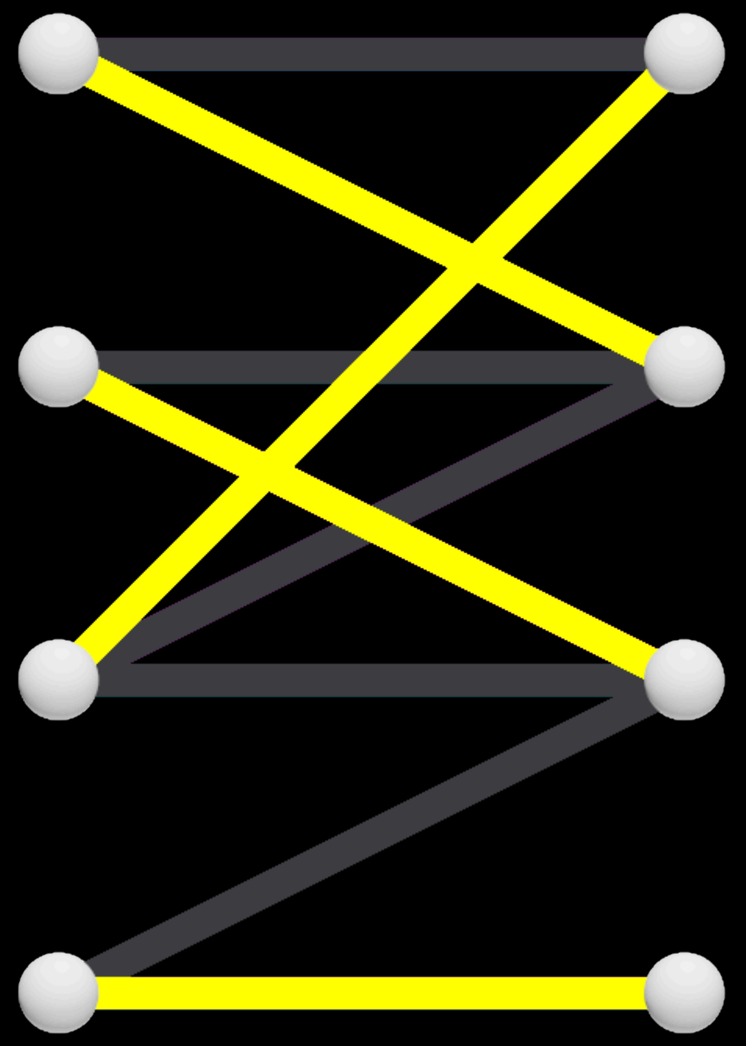

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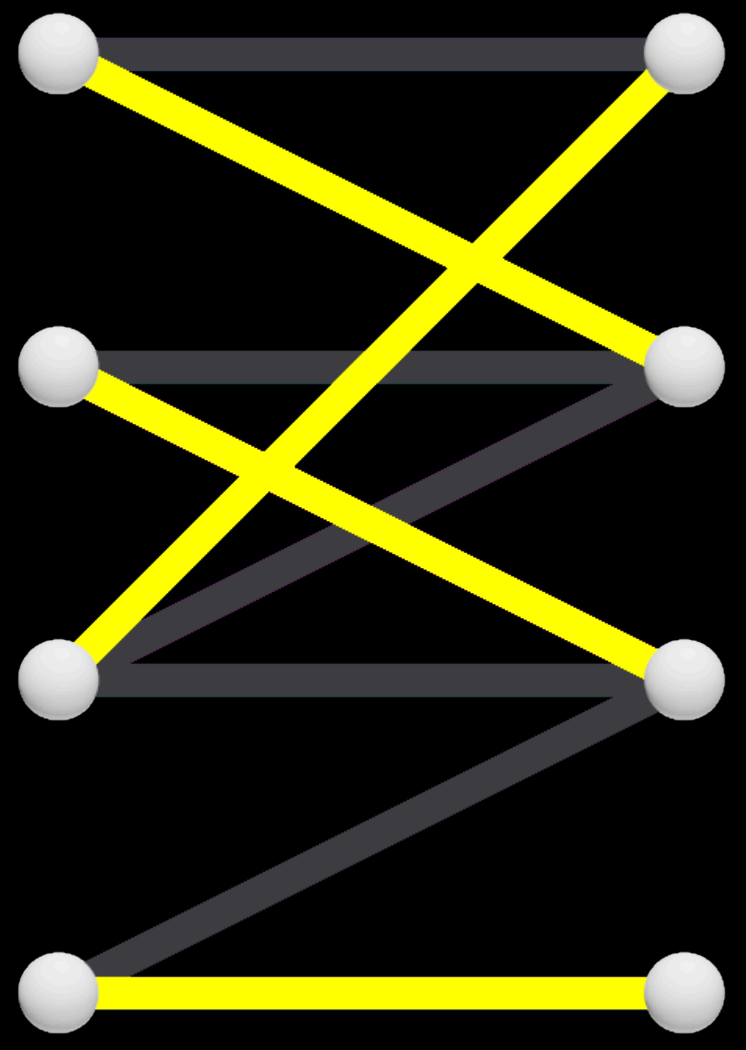

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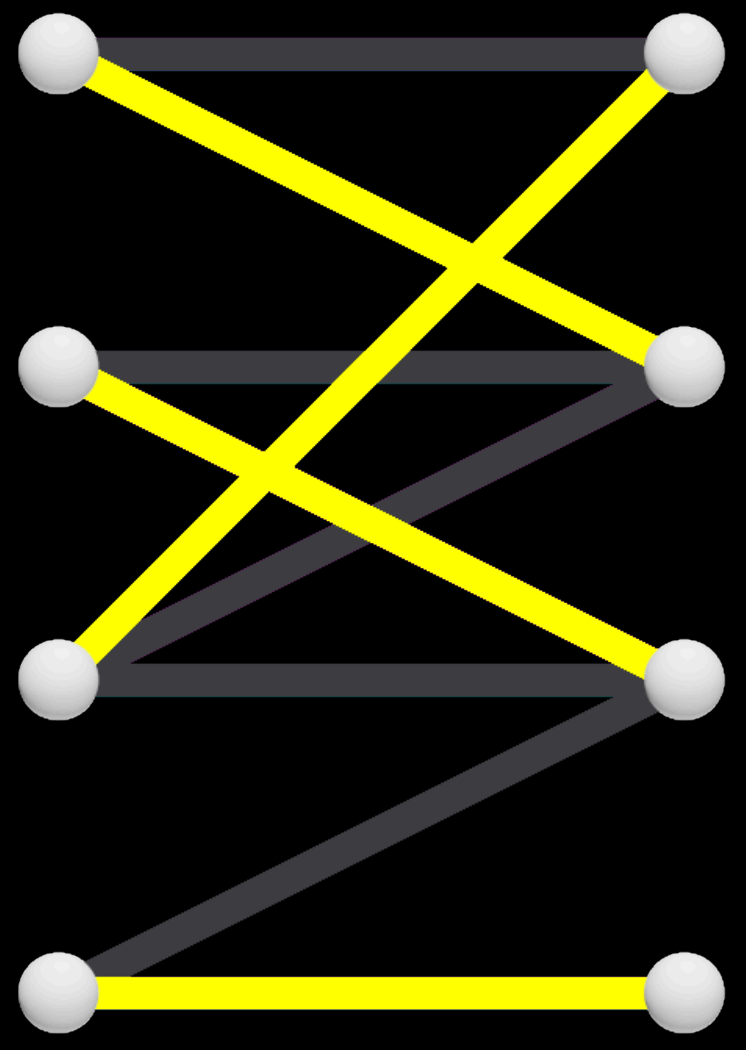

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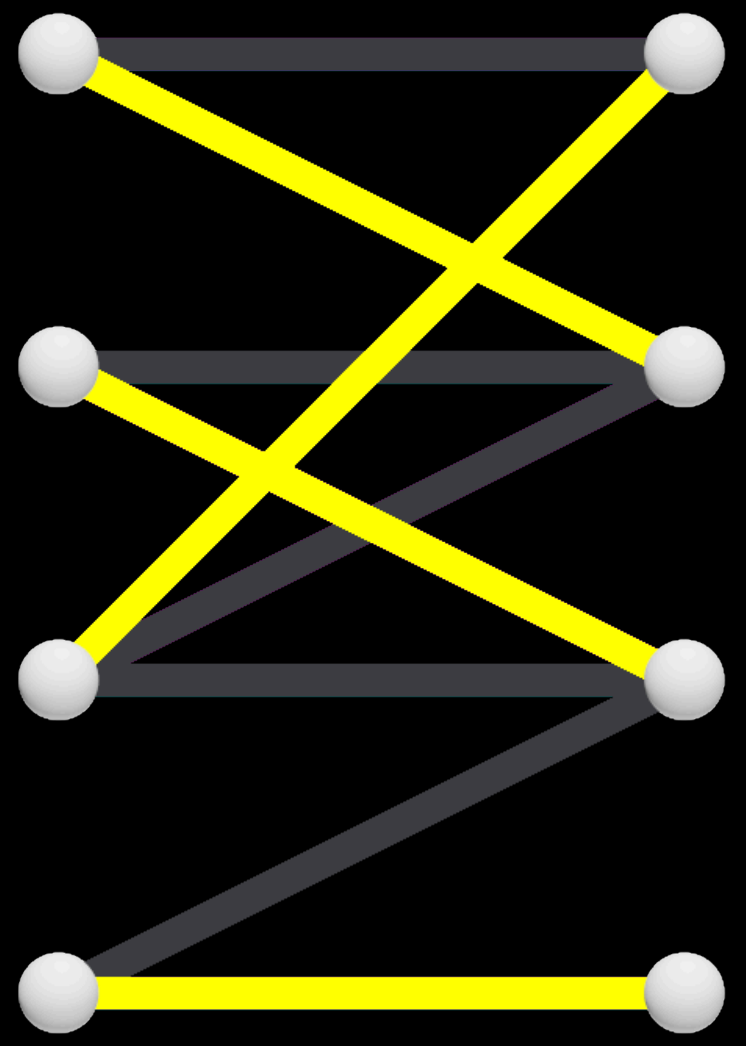

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- Even though the decision problem is in P !

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If $A \in \mathbb{R}^{n \times n}$ is a positive, *doubly stochastic* matrix, then
$$\text{per}(A) \geq \frac{n!}{n^n}$$

$$\frac{1}{n} J_n = \begin{bmatrix} 1/n & 1/n & \dots & 1/n \\ 1/n & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 1/n & \dots & \dots & 1/n \end{bmatrix}$$

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- Idea: If there exist L, R with LAR doubly stochastic then

$$\text{per}(A) = \frac{\text{per}(LAR)}{\text{per}(L)\text{per}(R)} \geq \frac{n!}{\text{per}(L)\text{per}(R)n^n}$$

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Max-flow-min-cut formulation

Sinkhorn-Knopp algorithm

[Linial,Samorodnitsky, Wigderson '04] : The first deterministic polynomial time approximation algorithm for the permanent.

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Commutative groups stabilize a flag:

$$0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_n = \mathbb{C}^n.$$

Representations of reductive groups are completely reducible:

$$V = W_1 \oplus W_2 \oplus \ldots \oplus W_m$$

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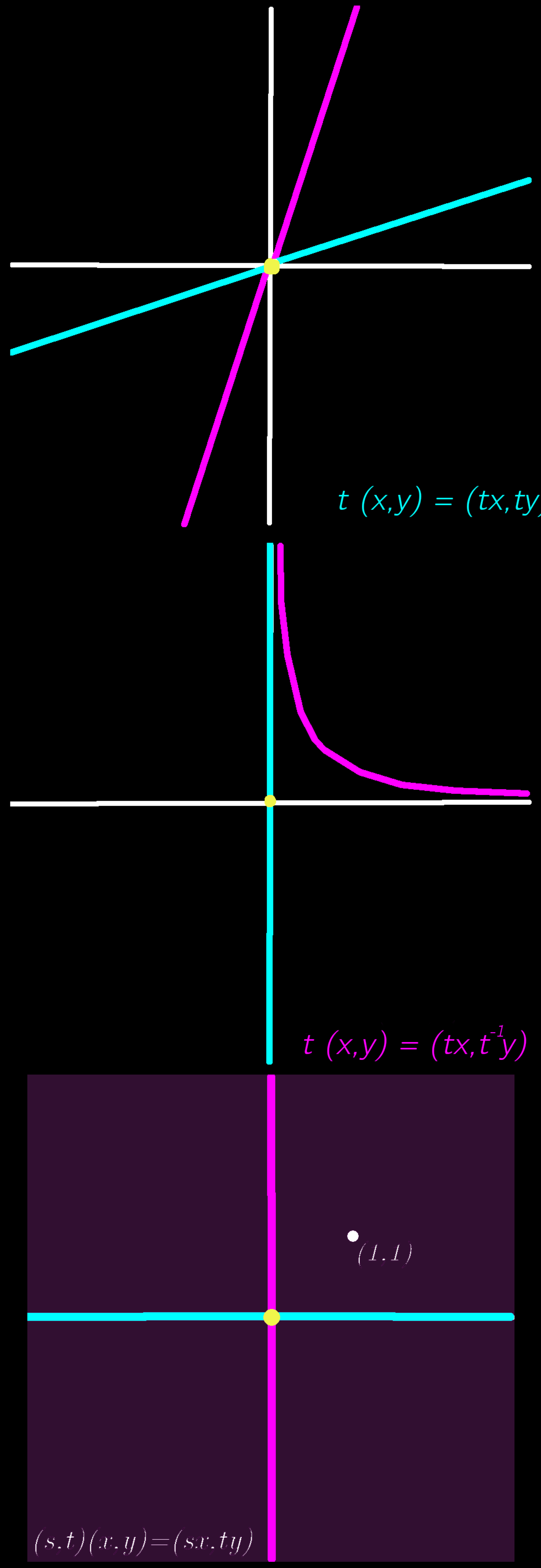

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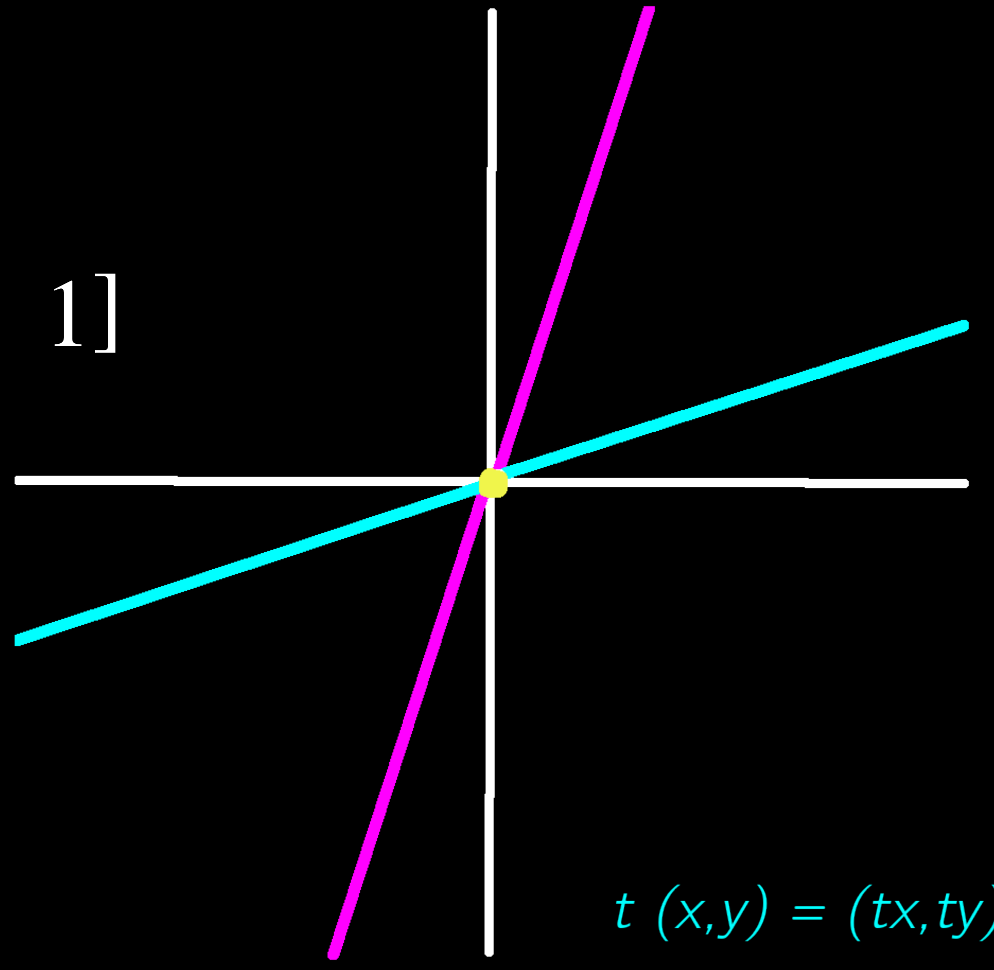
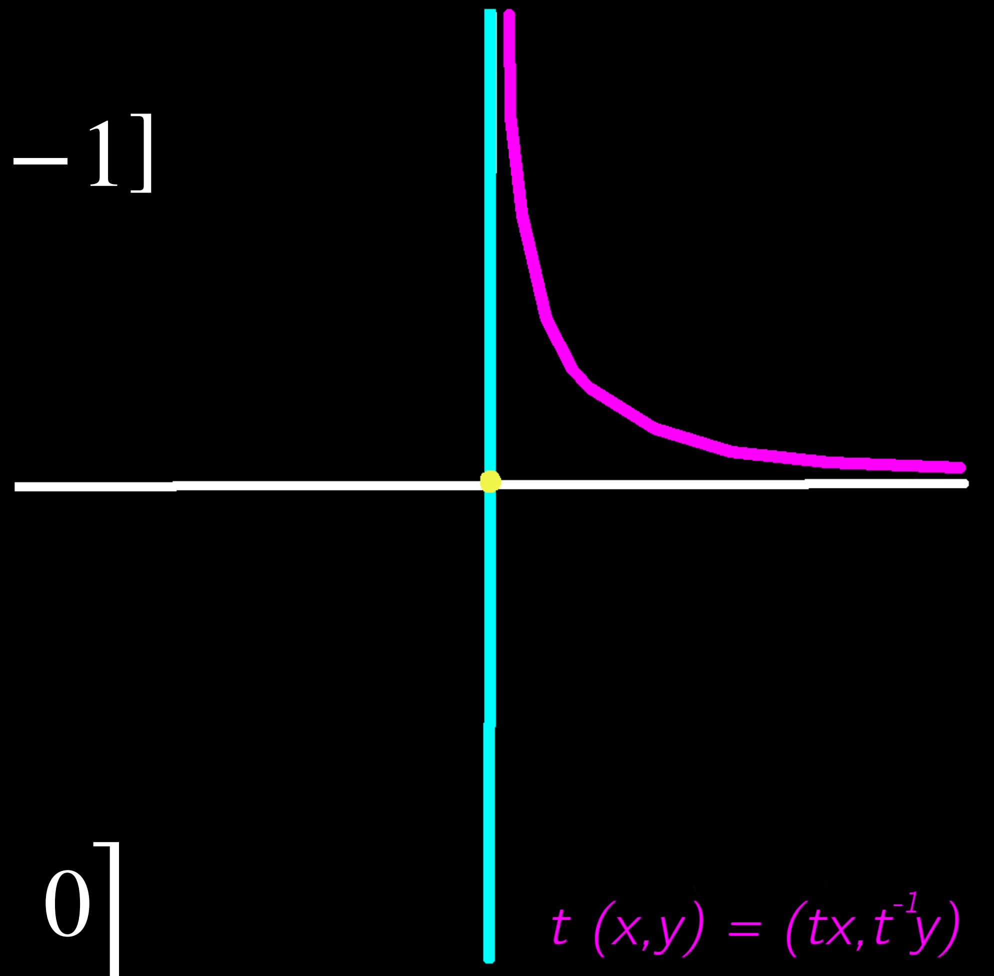
- **Definition:** The *weight matrix* of $(\mathbb{C}^{\times})^d \curvearrowright \mathbb{C}^n$ is the integer matrix $M \in \mathbb{Z}^{d \times n}$ having ω_i as its i-th column.

$M = [1 \quad 1]$



$M = [1 \quad -1]$



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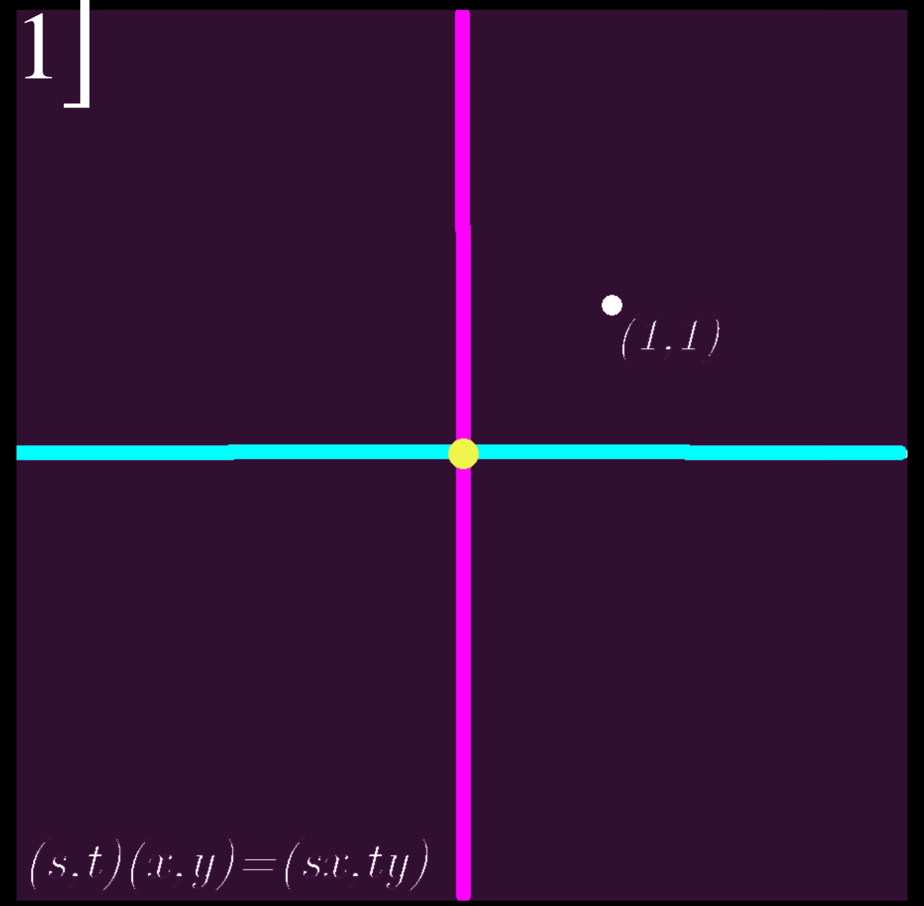

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- Capacity:
  $$\mathrm{cap}(v) := \inf_{t \in T} \|tv\|_2$$

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Note: In the case of matrix scaling,
Nullcone membership ~ is the matrix scalable?
Orbit closure containment problem ~ is the matrix almost scalable?
Minimizing ∇F_v ~ $(\mathbf{r}, \mathbf{c})-$scaling.

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~ entropy maximization (D_{KL} is the Shannon entropy when $q = \mathbf{1}$)

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- Ongoing research for non-commutative groups: [Bürgisser, Franks, Garg, Oliveira, Walter, Wigderson '19]

- Theorem (Bürgisser, D., Makam, Walter, Wigderson):
  Let $M \in \mathbb{Z}^{d \times n}$ be the weight matrix of $(\mathbb{C}^\times)^d \curvearrowright \mathbb{C}^n$ and suppose $v, w \in \mathbb{Q}(i)^n$. Let b denote the maximum of the bit-lengths of the entries of v, w and M. Then, in poly(d, n, b) time we can decide orbit equality, orbit closure intersection and orbit closure containment.

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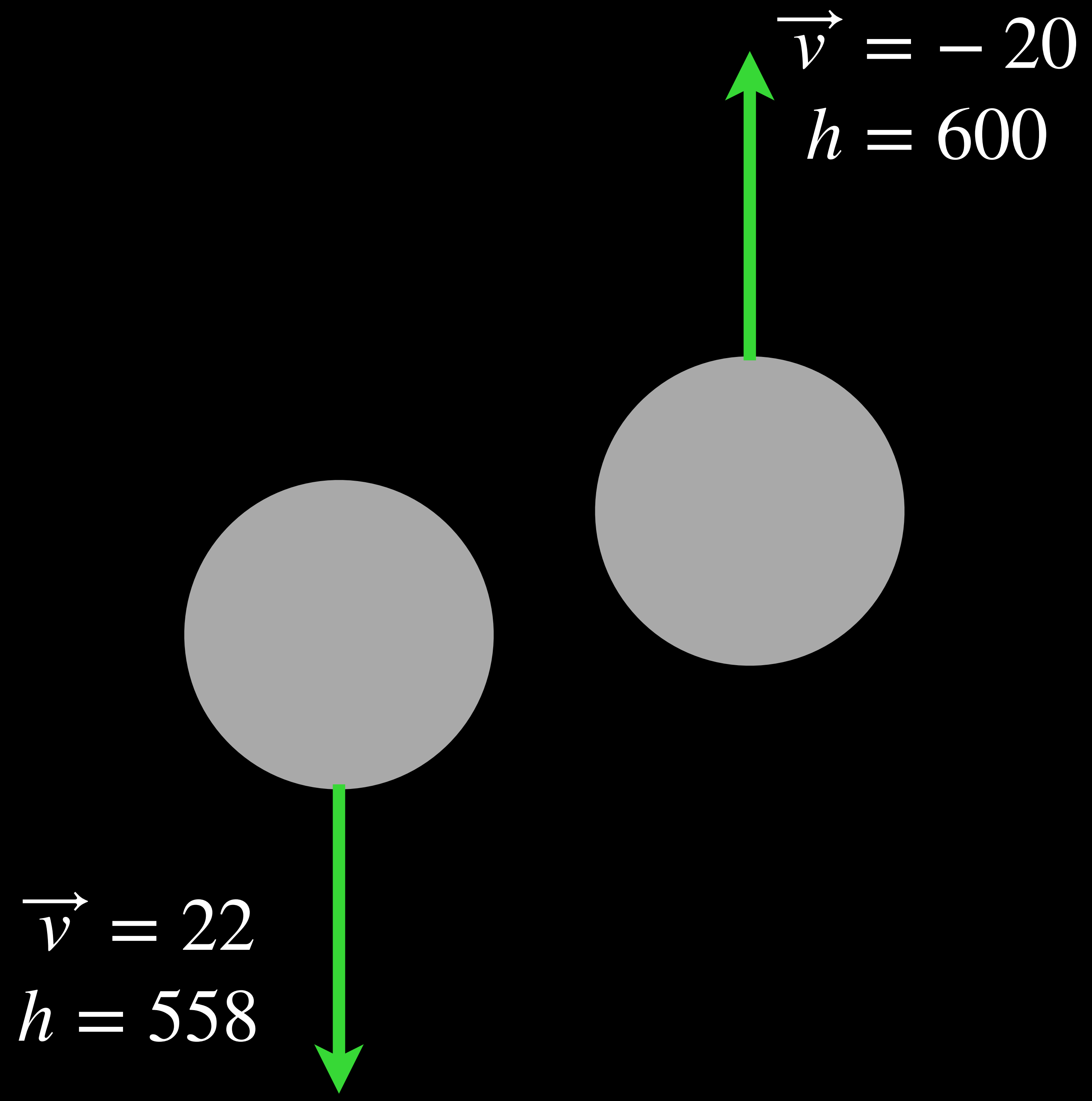
- Orbit closure intersection problem admits polynomial time algorithm for:
  Left-right action, [Derksen,Makam '18], [Allen-Zhu, Garg, Li, Oliveira, Wigderson '18], [Ivanyos, Qiao, Subrahmanyam '18]
  simultaneous conjugation, [Derksen, Makam '18]
  quiver representations,
  actions of groups of bounded dimension [Mulmuley '12]
  …



$\vec{v} = -20$
$h = 600$
$\vec{v} = 22$
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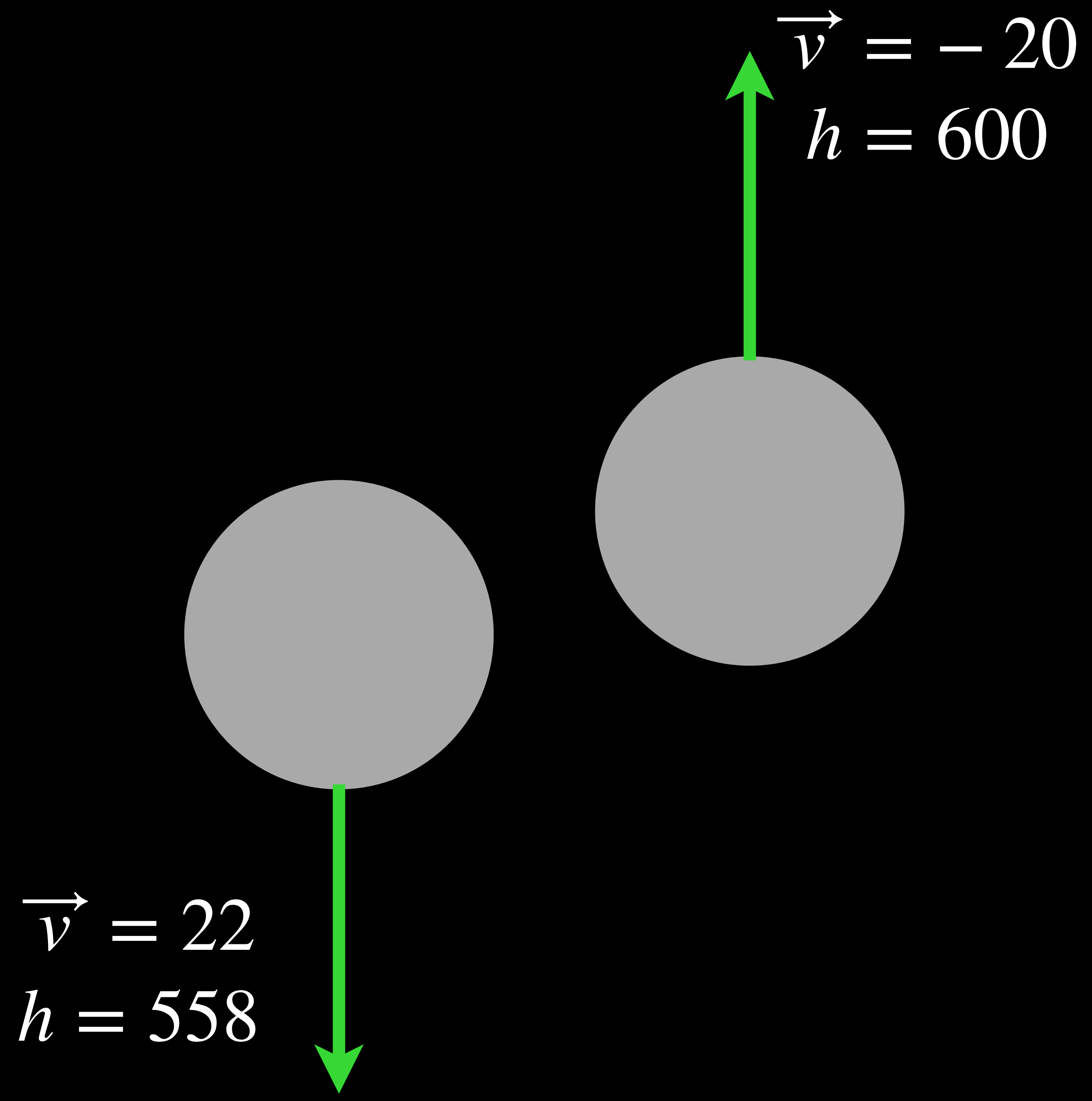
- How to tell if two shapshots belong to the same ball?



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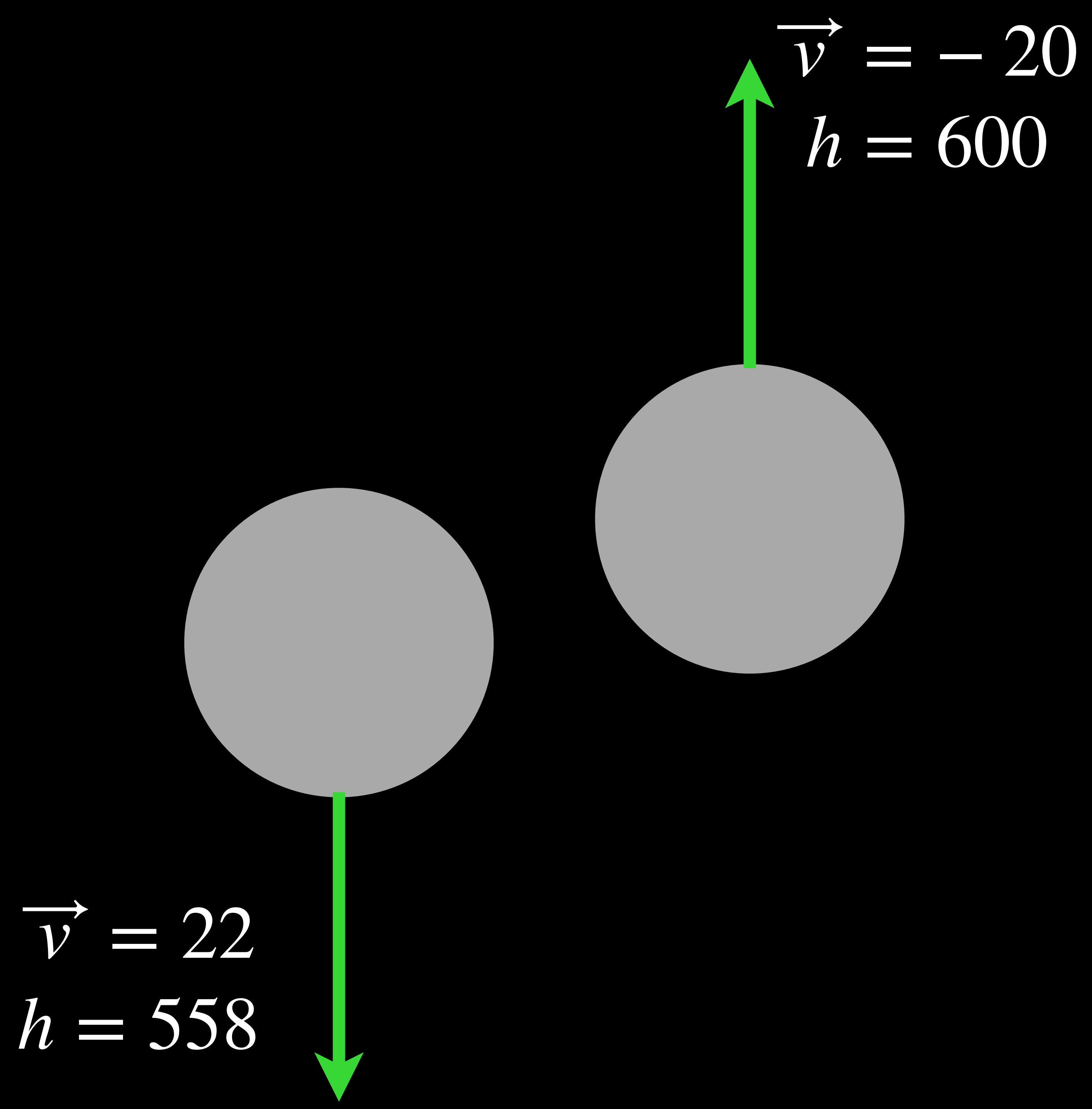

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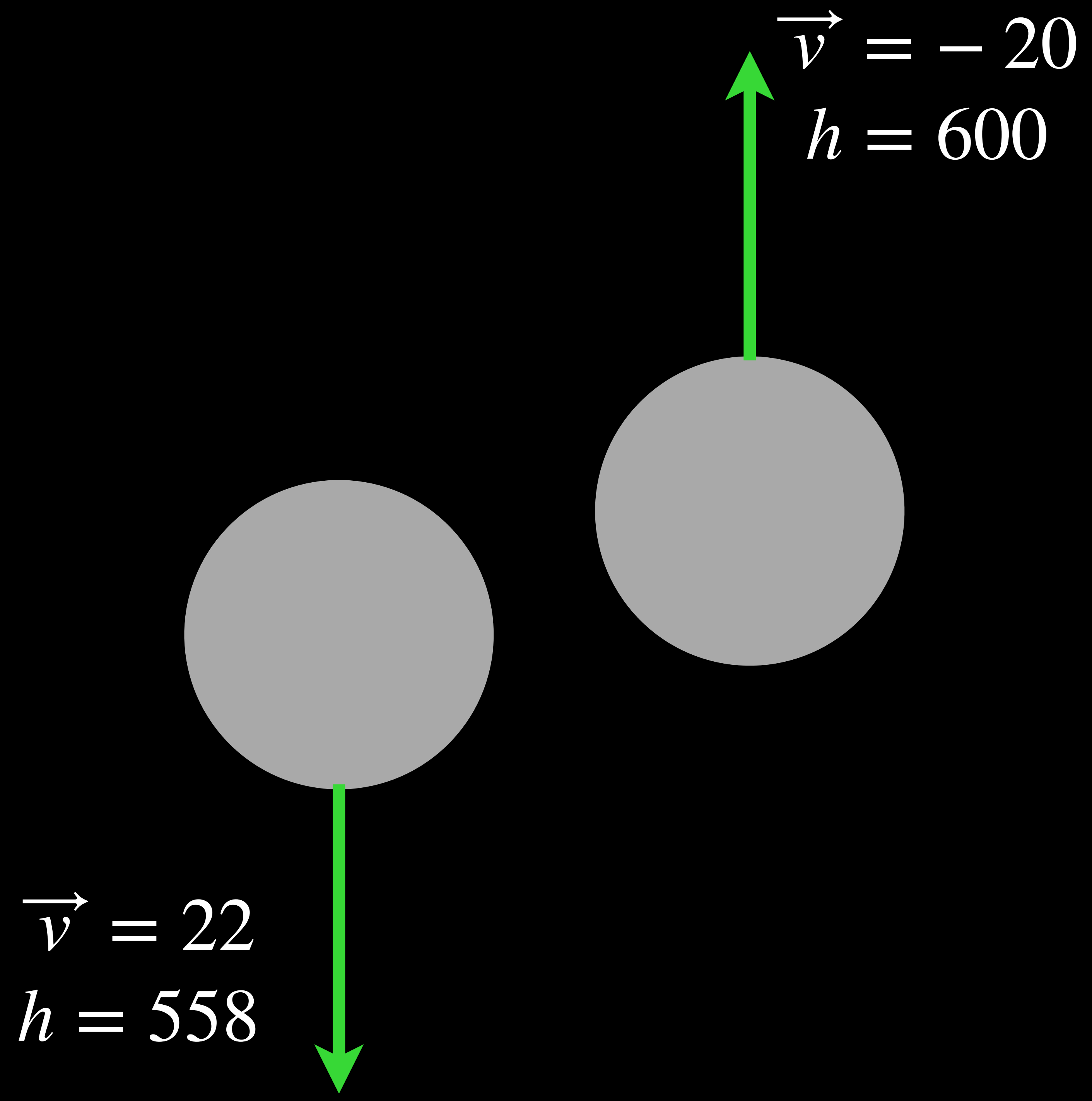

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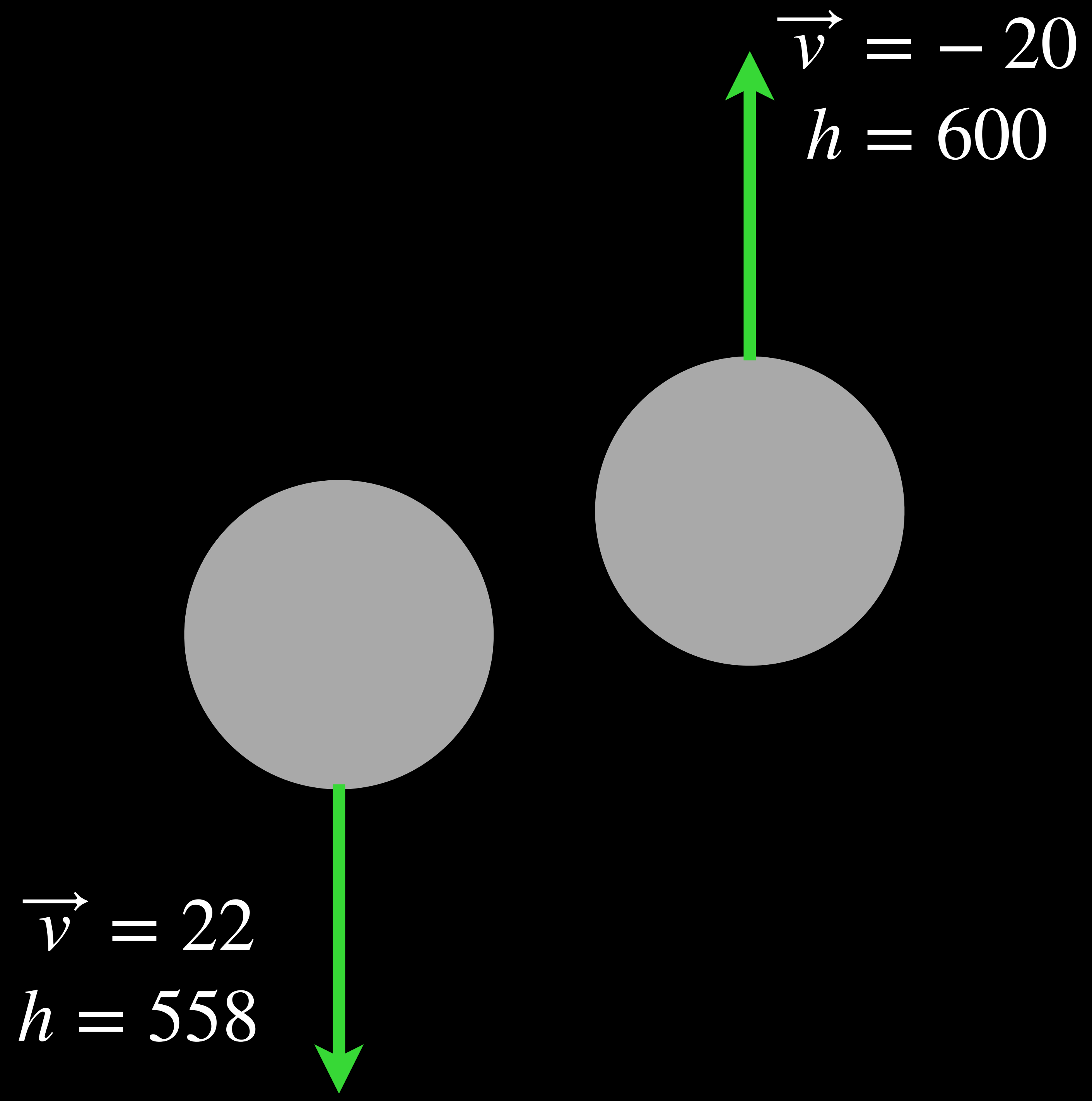

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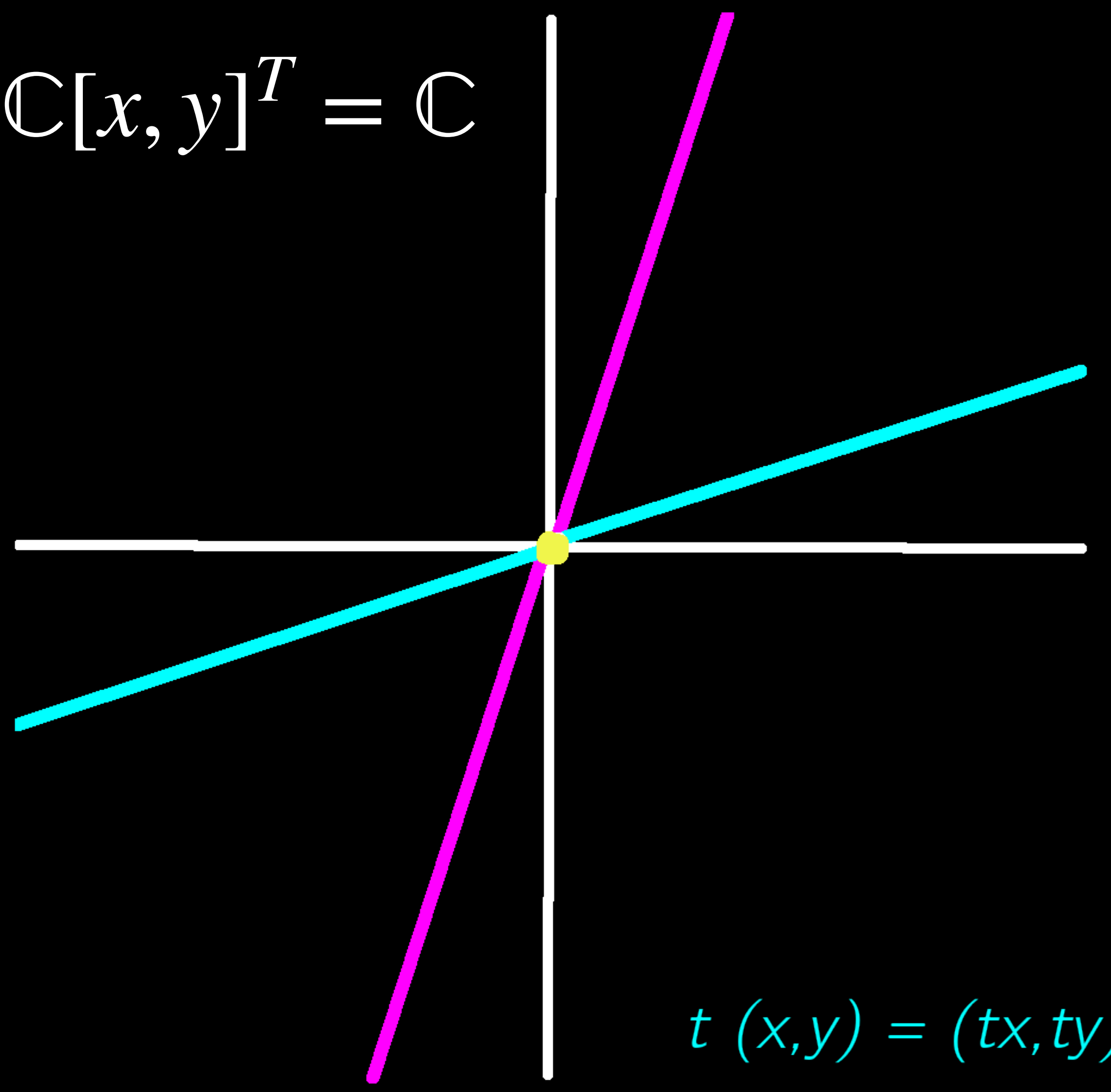
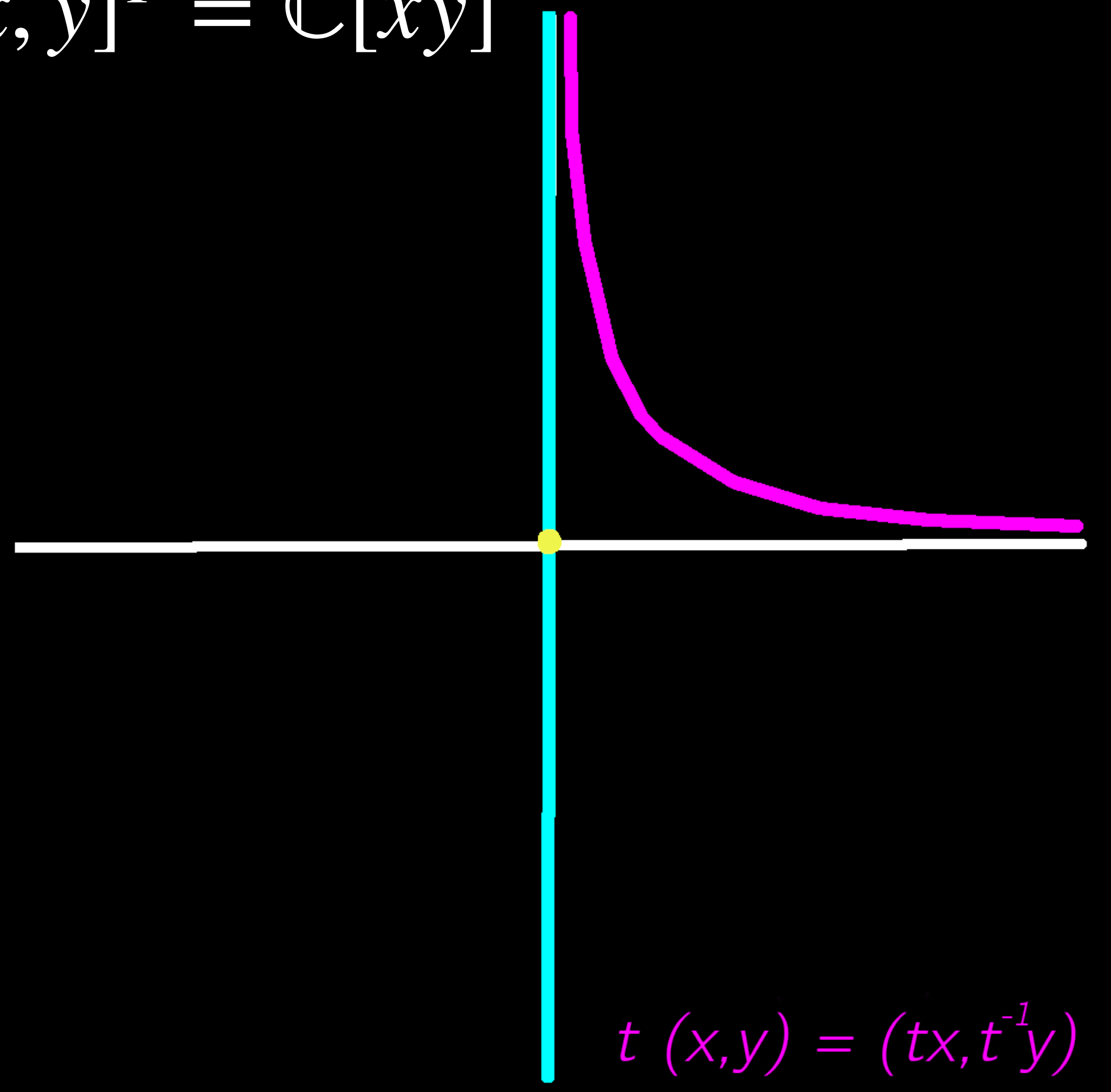


$\mathbb{C}[x, y]^T = \mathbb{C}[xy]$



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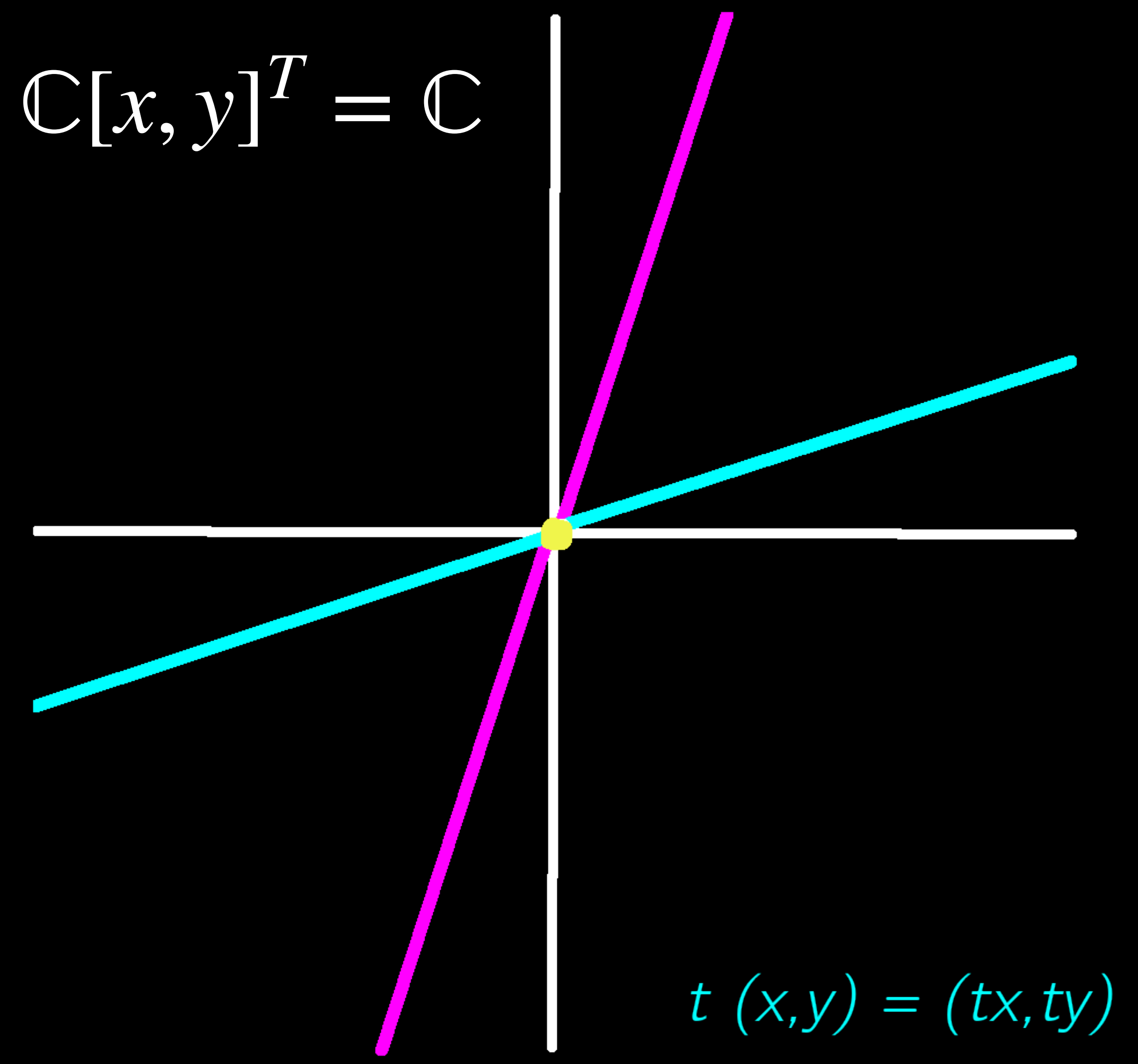

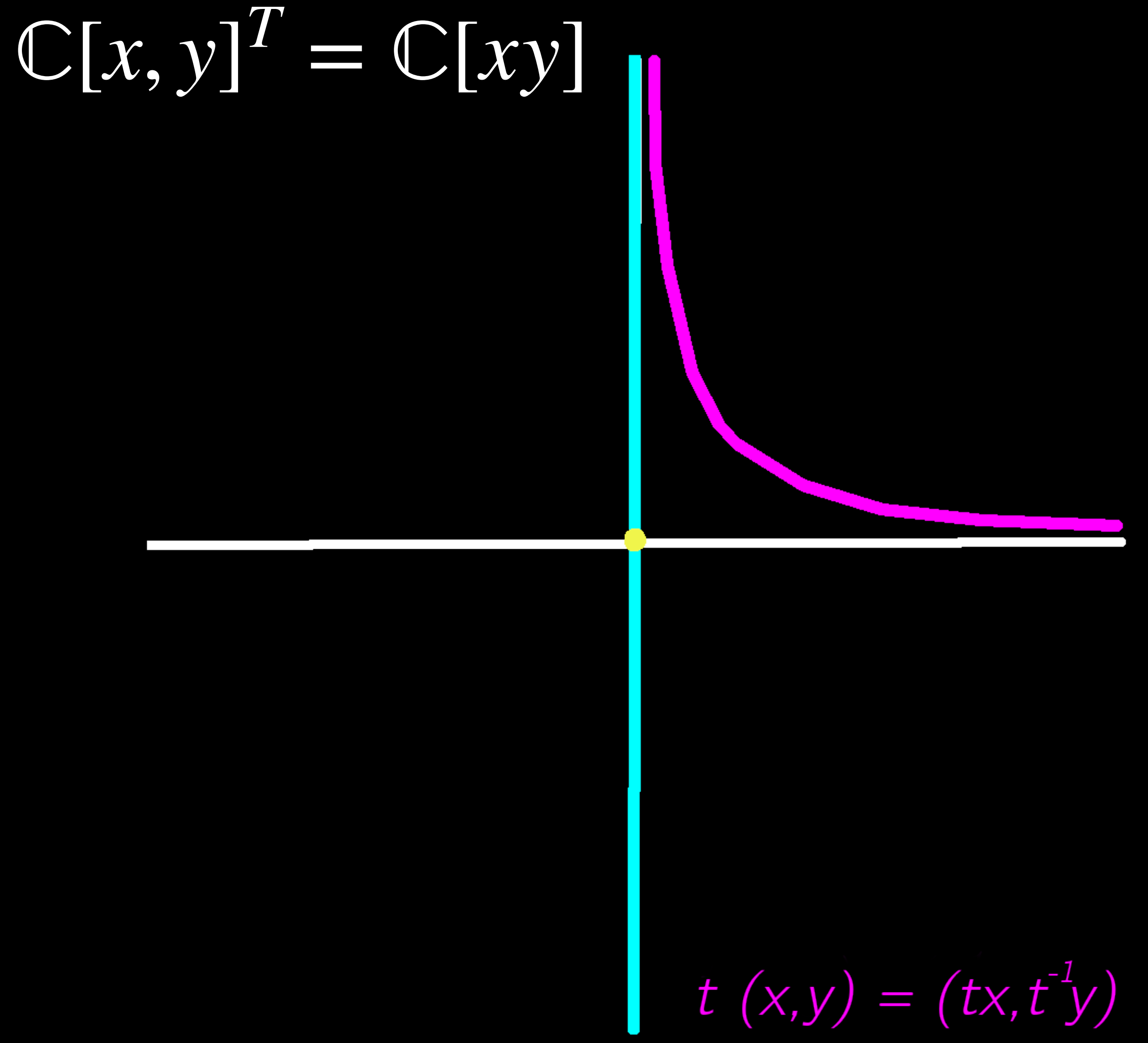

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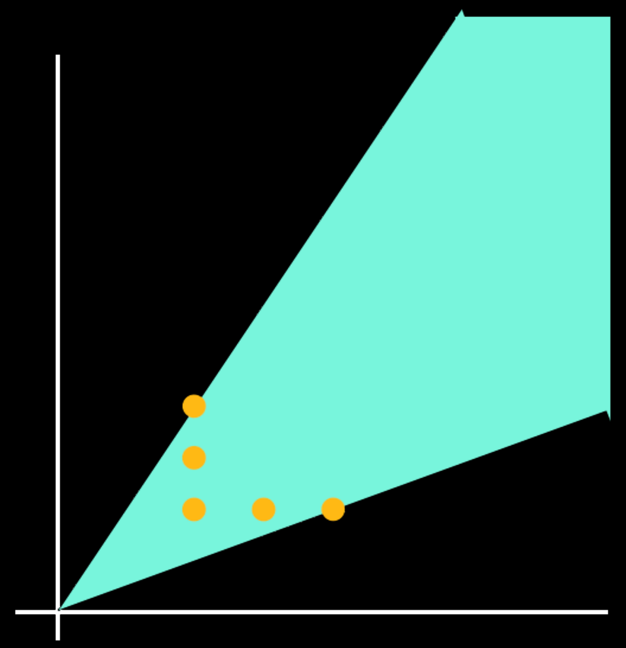

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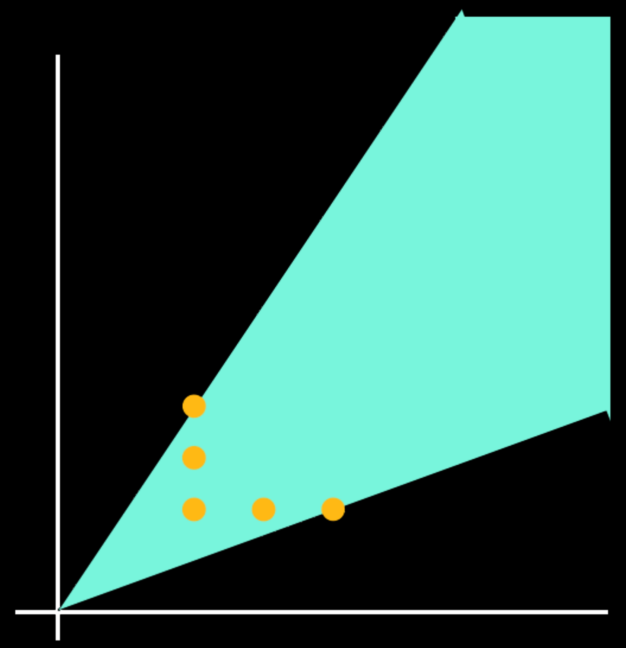

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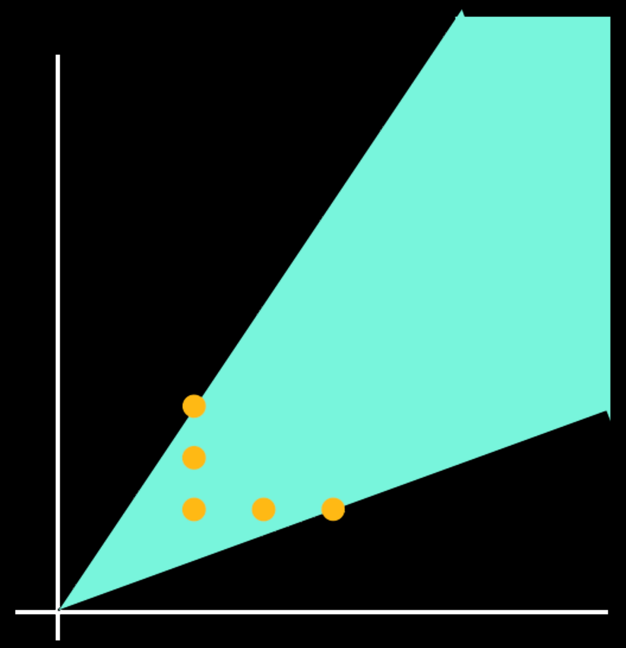

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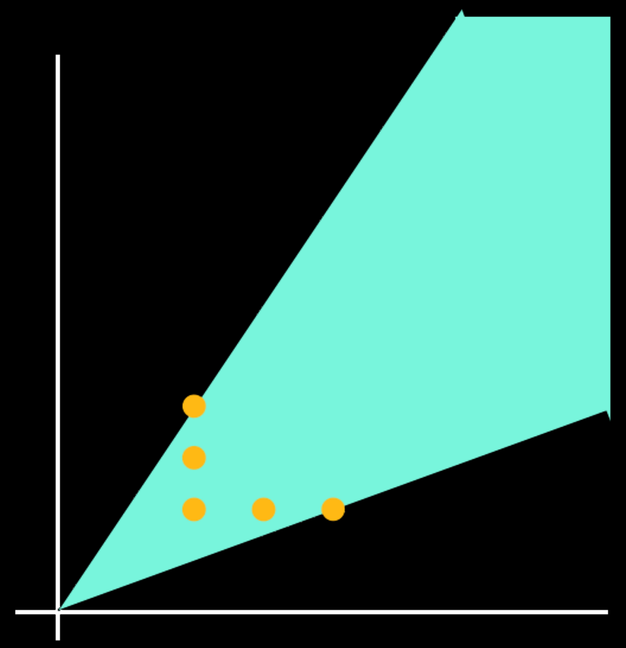

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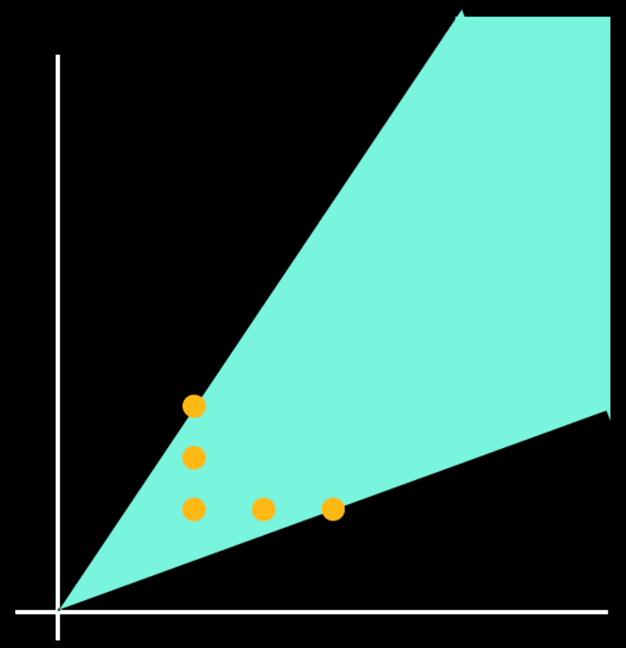
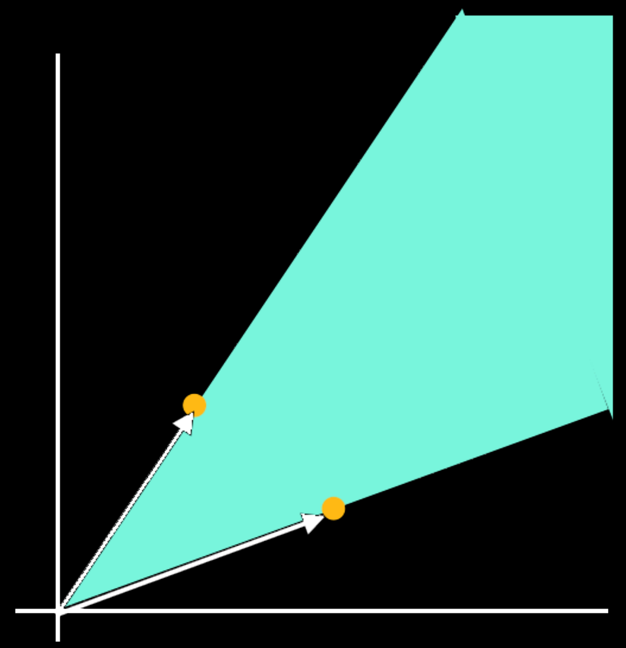
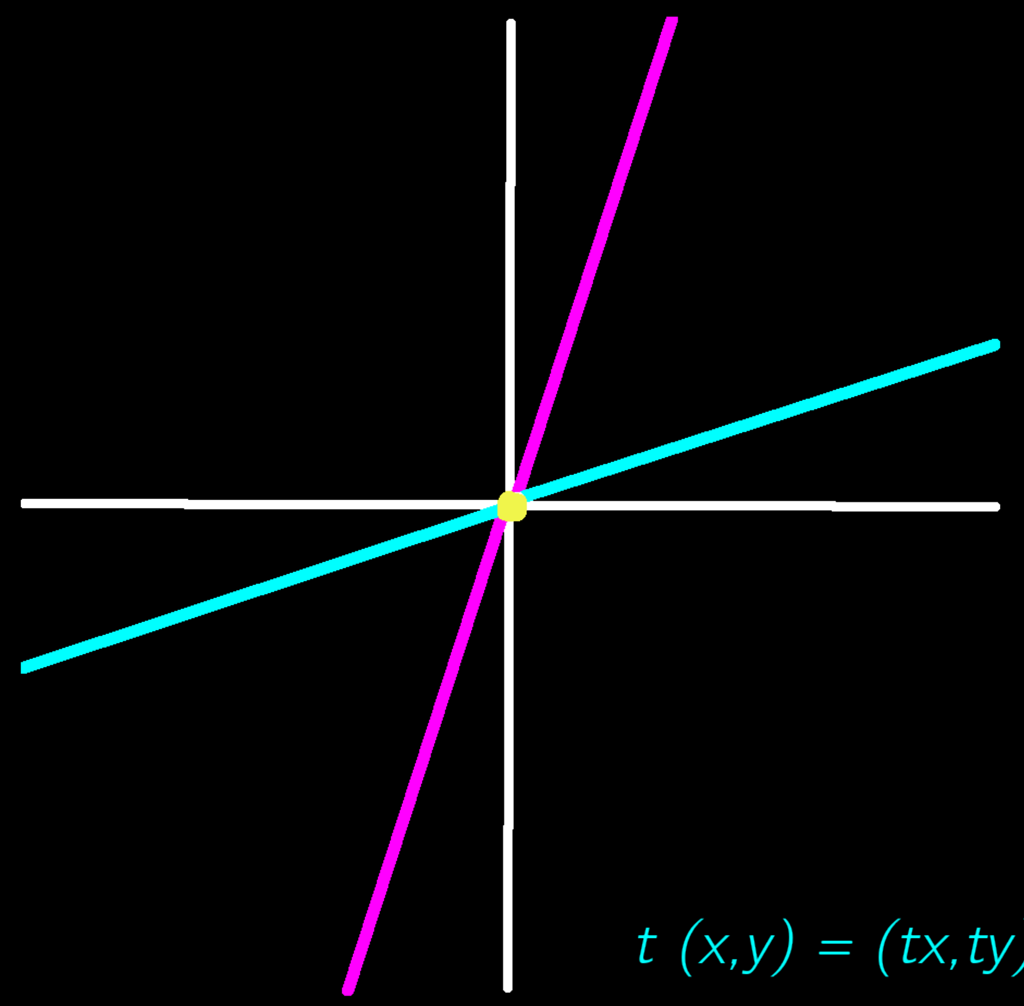


- Rational invariants: $\varphi = \frac{f}{g} \in \mathbb{C}(x_1, x_2, \ldots, x_n)$, $\varphi(tv) = \varphi(v)$ for every $t \in T, v \in V$.
- Spanned by invariant *Laurent monomials*: $\{x^\alpha \mid \alpha \in \mathbb{Z}^n, M\alpha = 0\}$
- Exponent vectors form a *lattice*: $L = \{\alpha \in \mathbb{Z}^n \mid M\alpha = 0\}$, which admits a *lattice basis*, $\mathscr{B}$.
- $\mathscr{B}$ has cardinality at most n.





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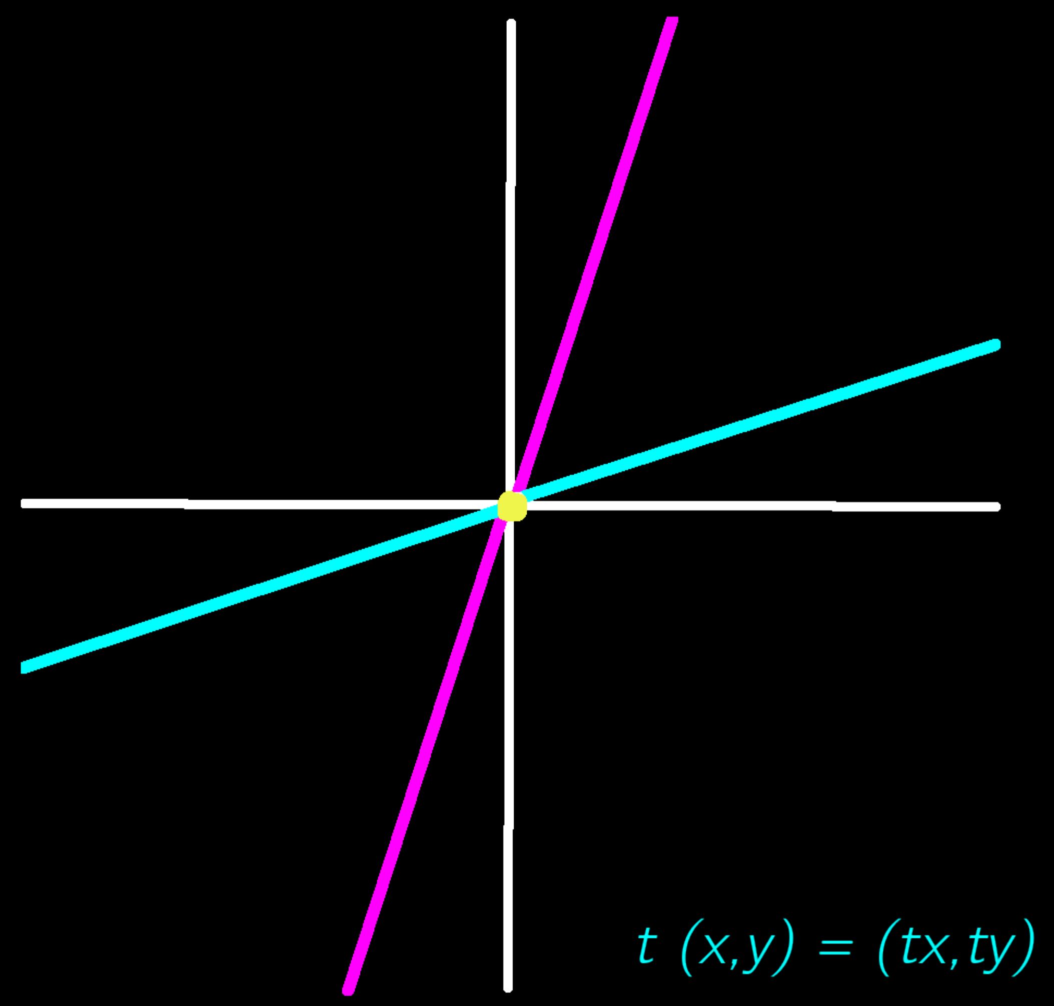

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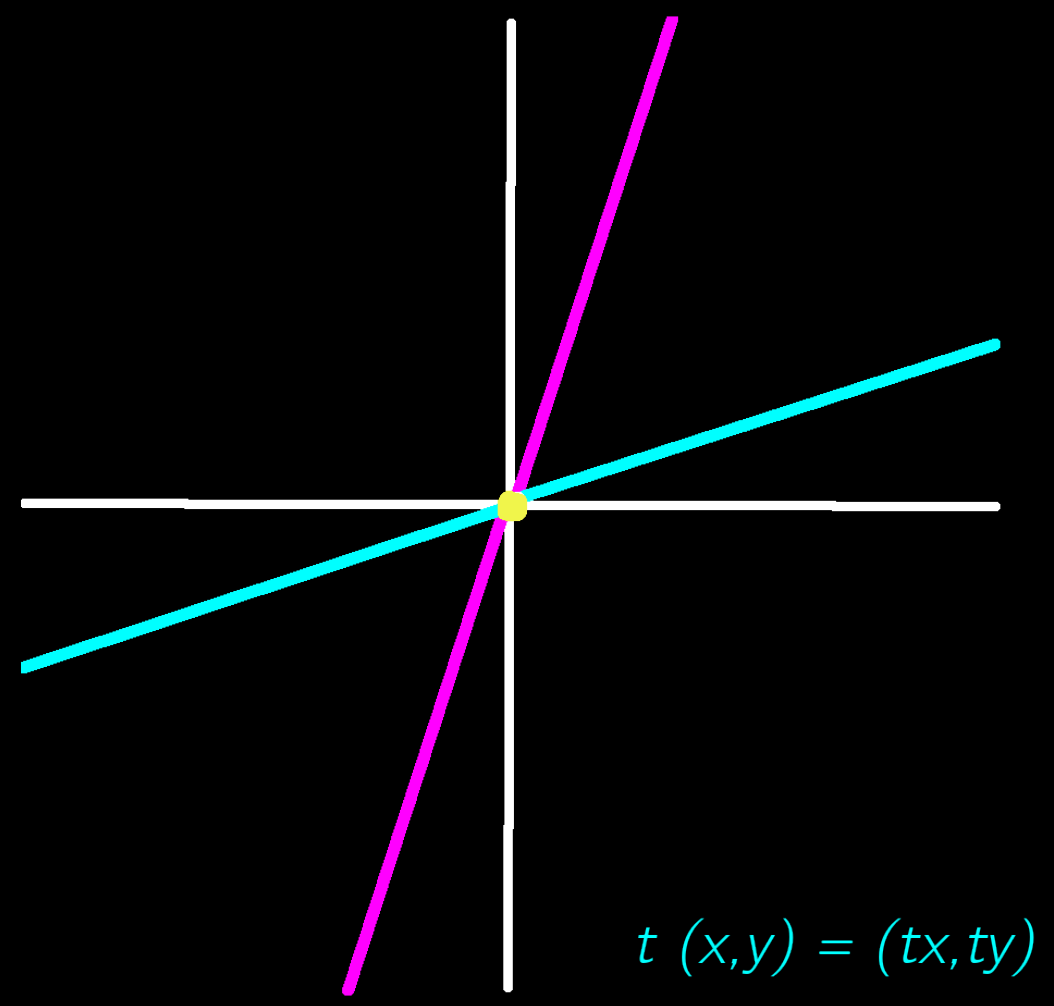
  Orbit closures intersect at 0.



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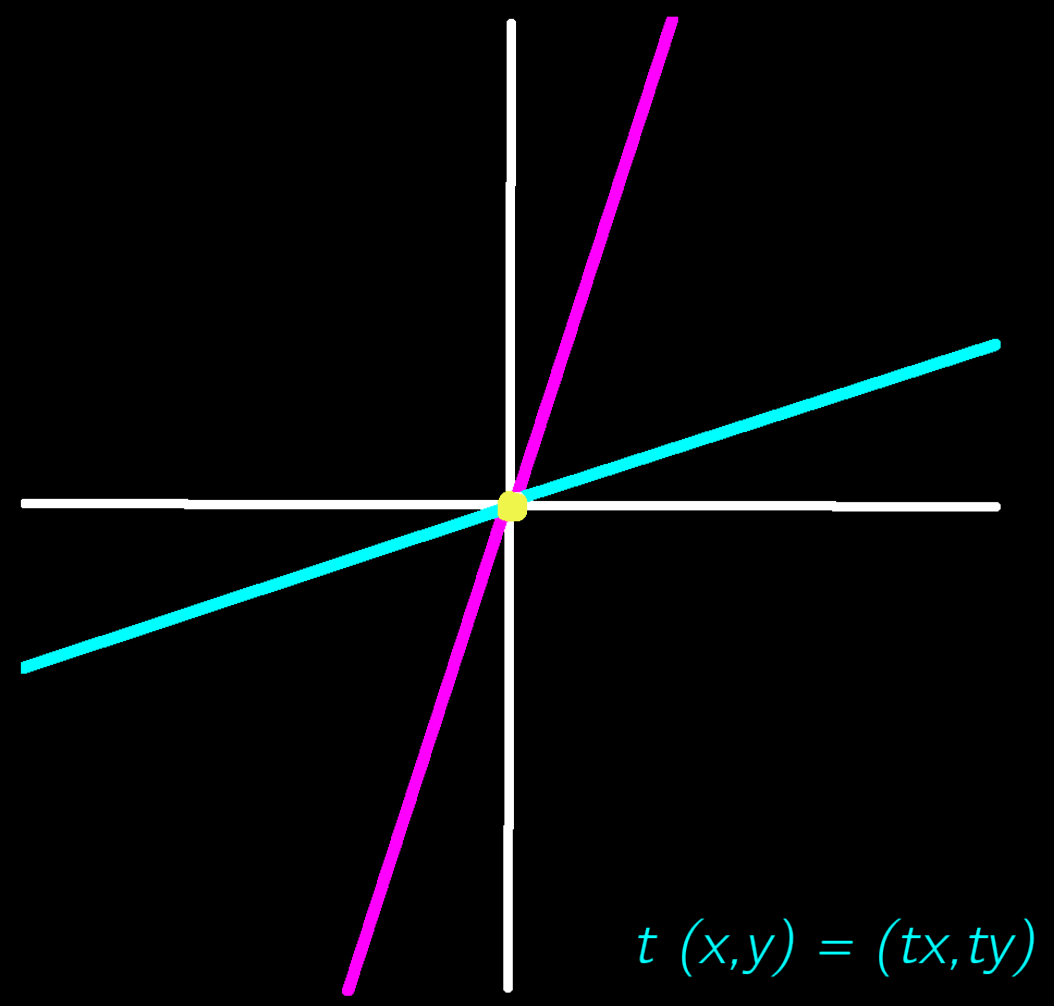

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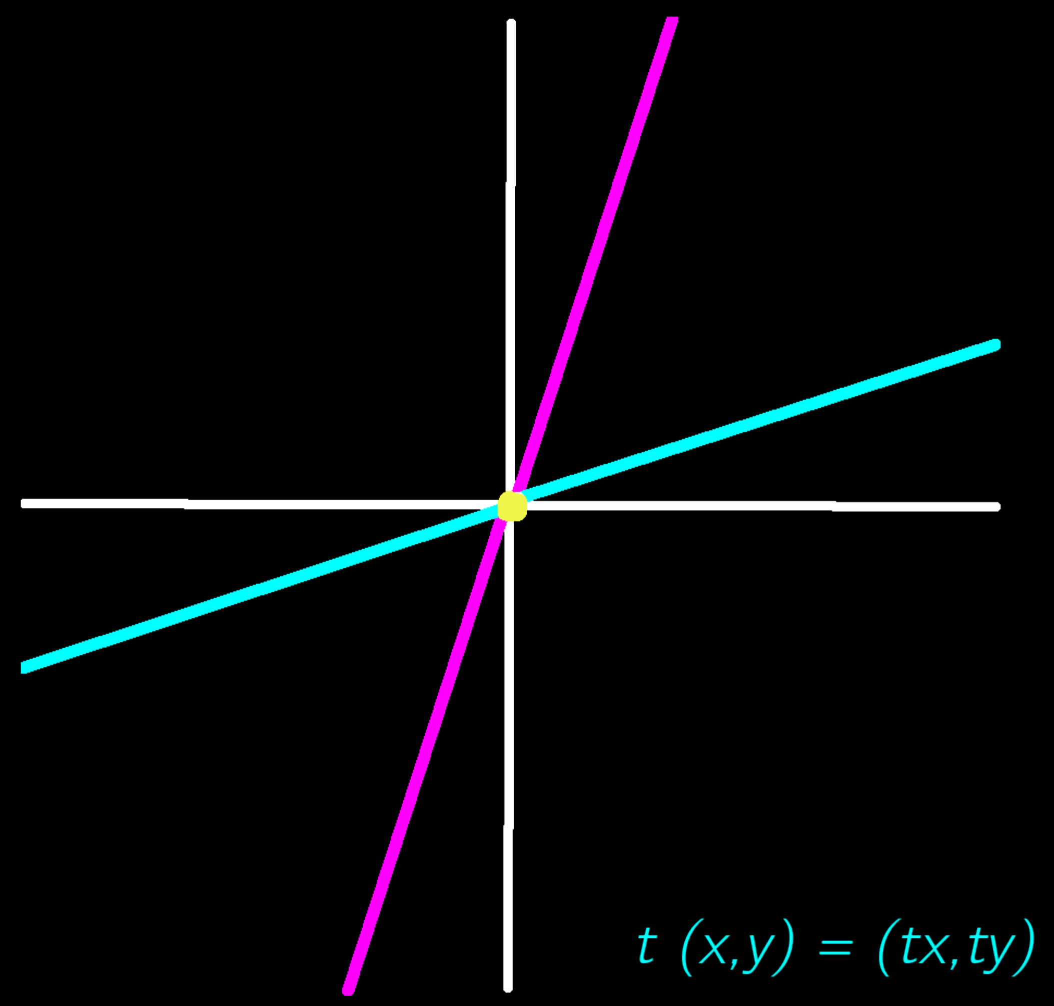

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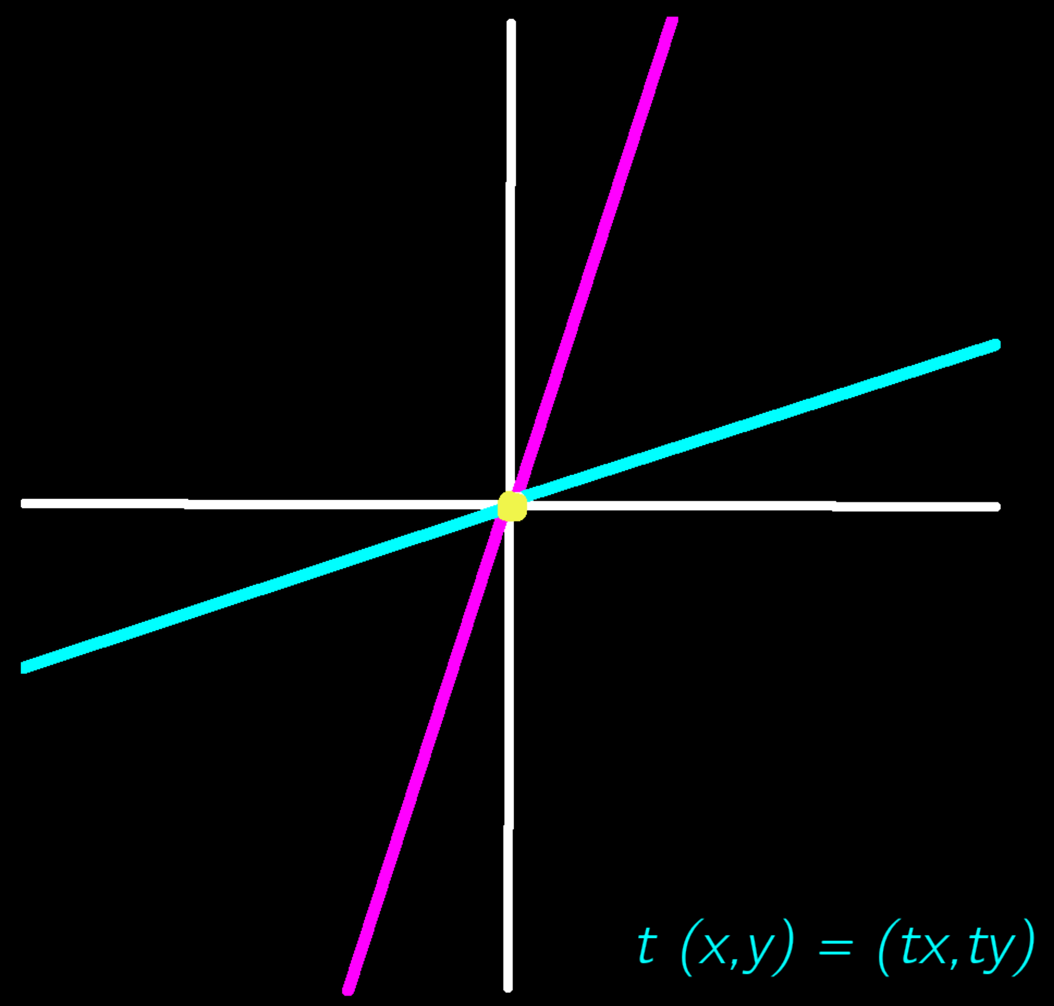

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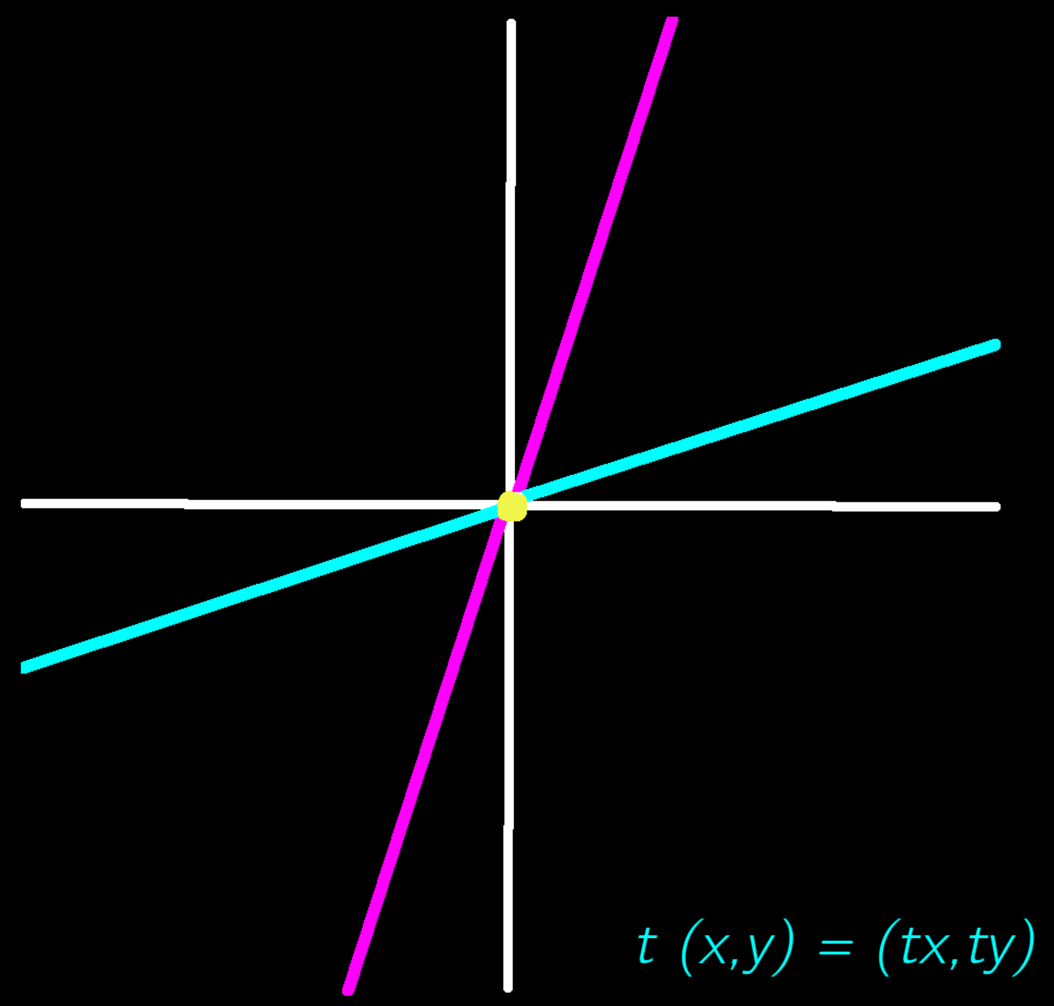

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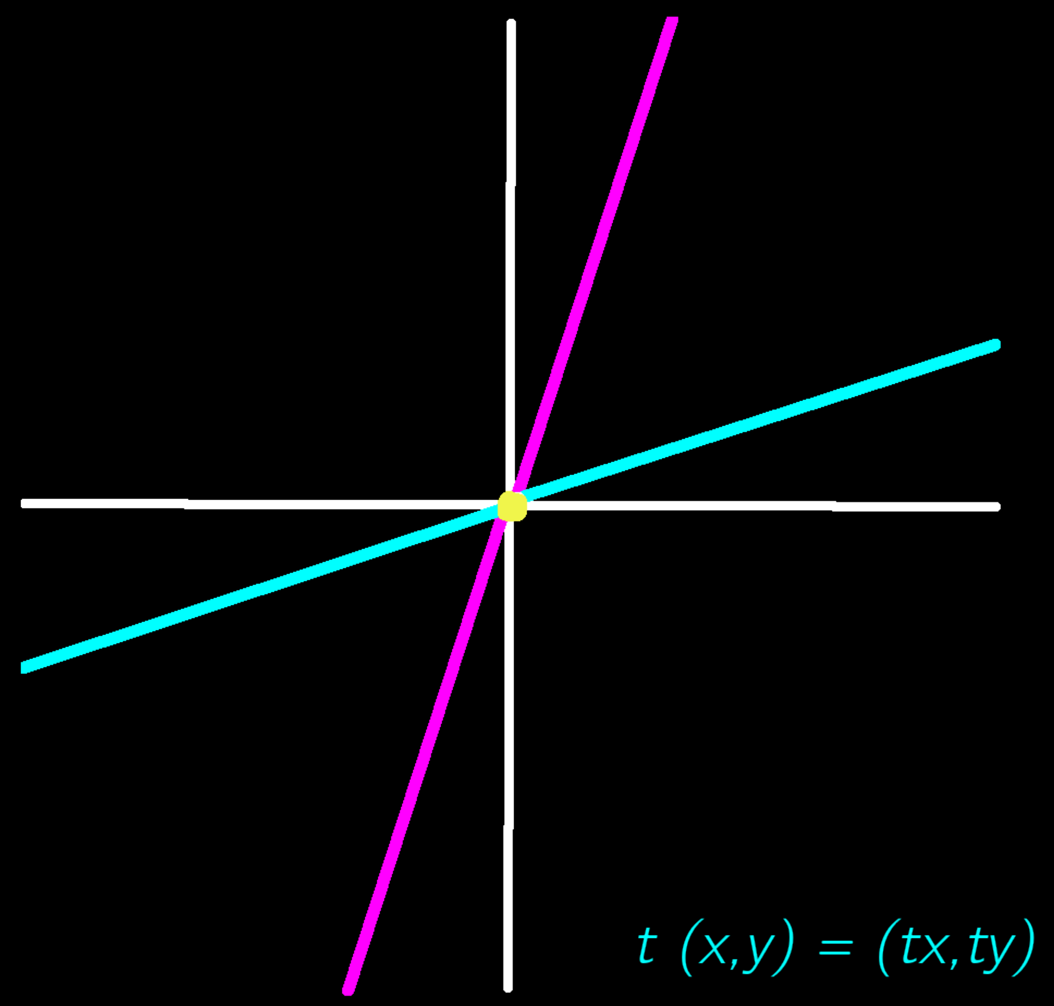

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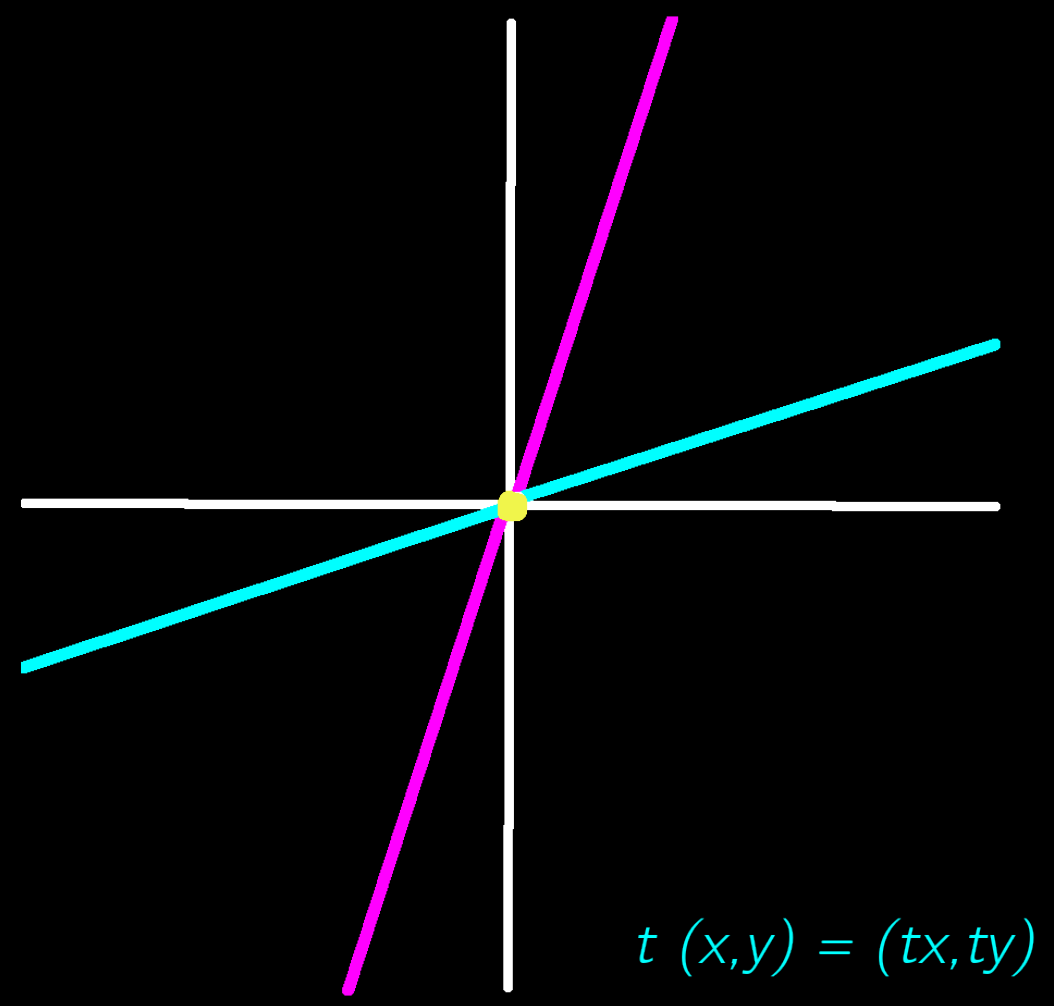

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- For $\alpha \in L$, check $x^\alpha(v) = x^\alpha(w)$. If all equal, $Tv = Tw$. If not, $Tv \neq Tw$.

Computing a lattice basis

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- As UMW is diagonal, a basis for the latter lattice is given by the columns $W^{(r+1)}, \ldots, W^{(n)}$ of W, where r is the rank of M.

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- Theorem (Ge '93): Laurent monomial equivalence over a number field K can be tested in polynomial time.

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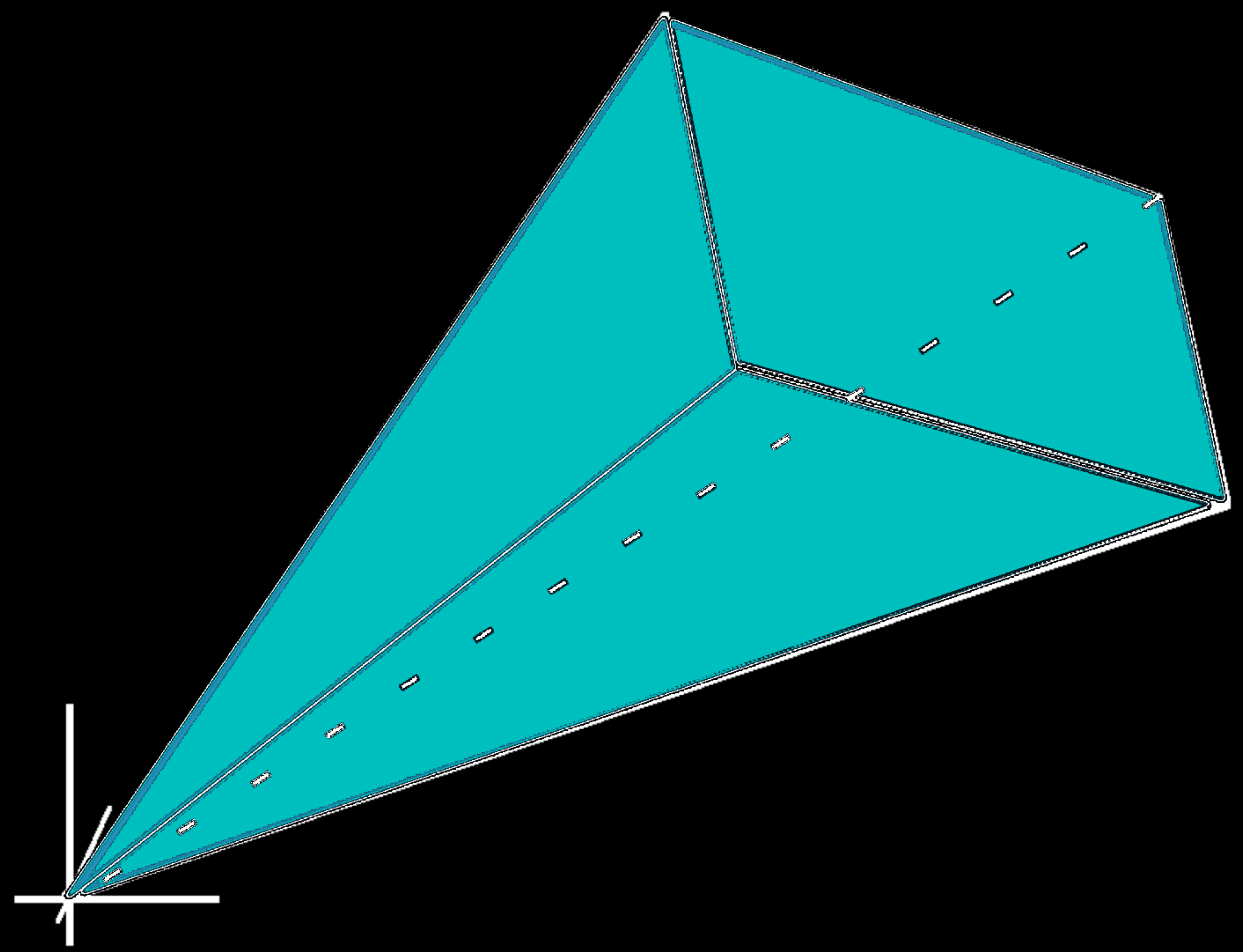

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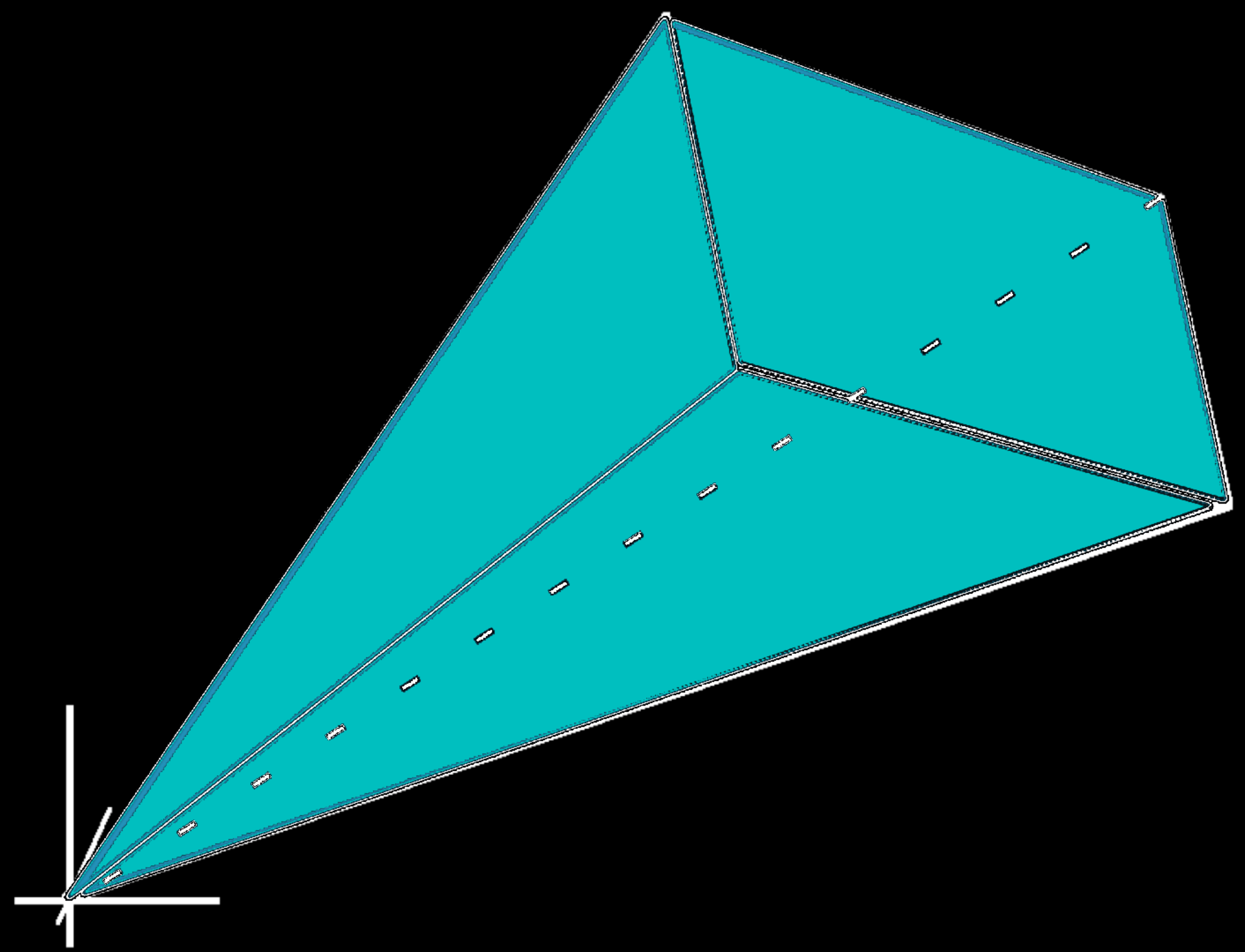

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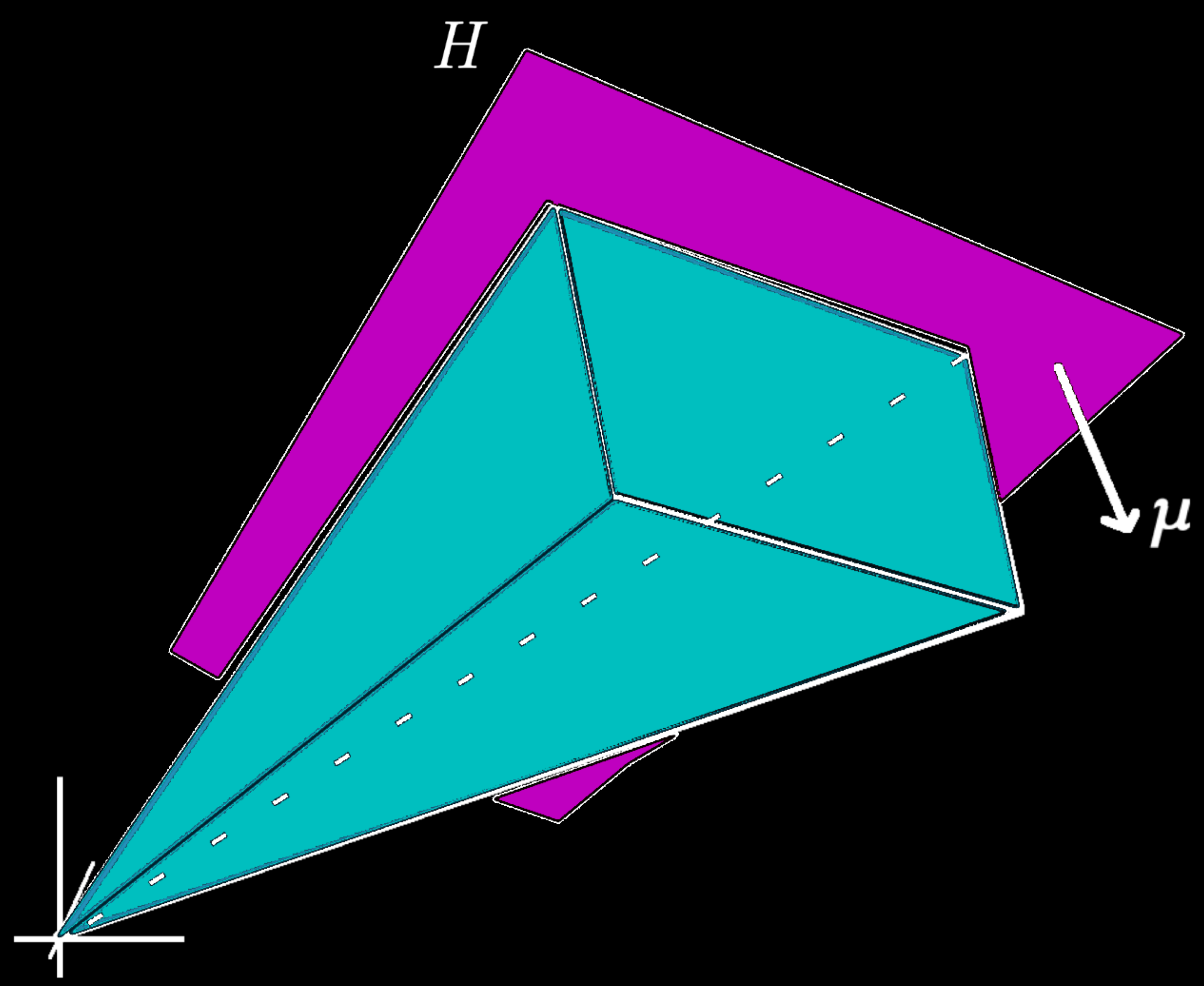

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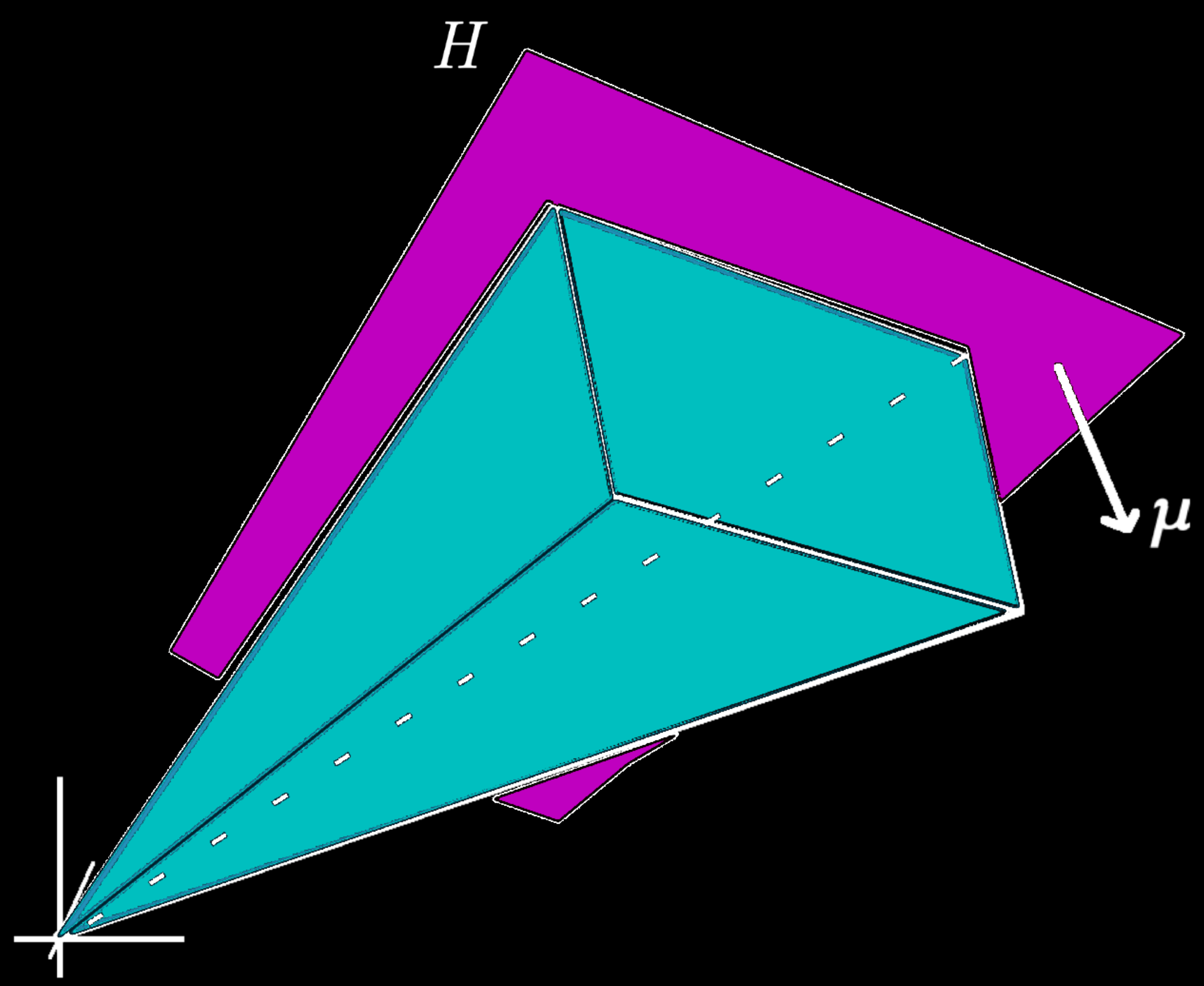

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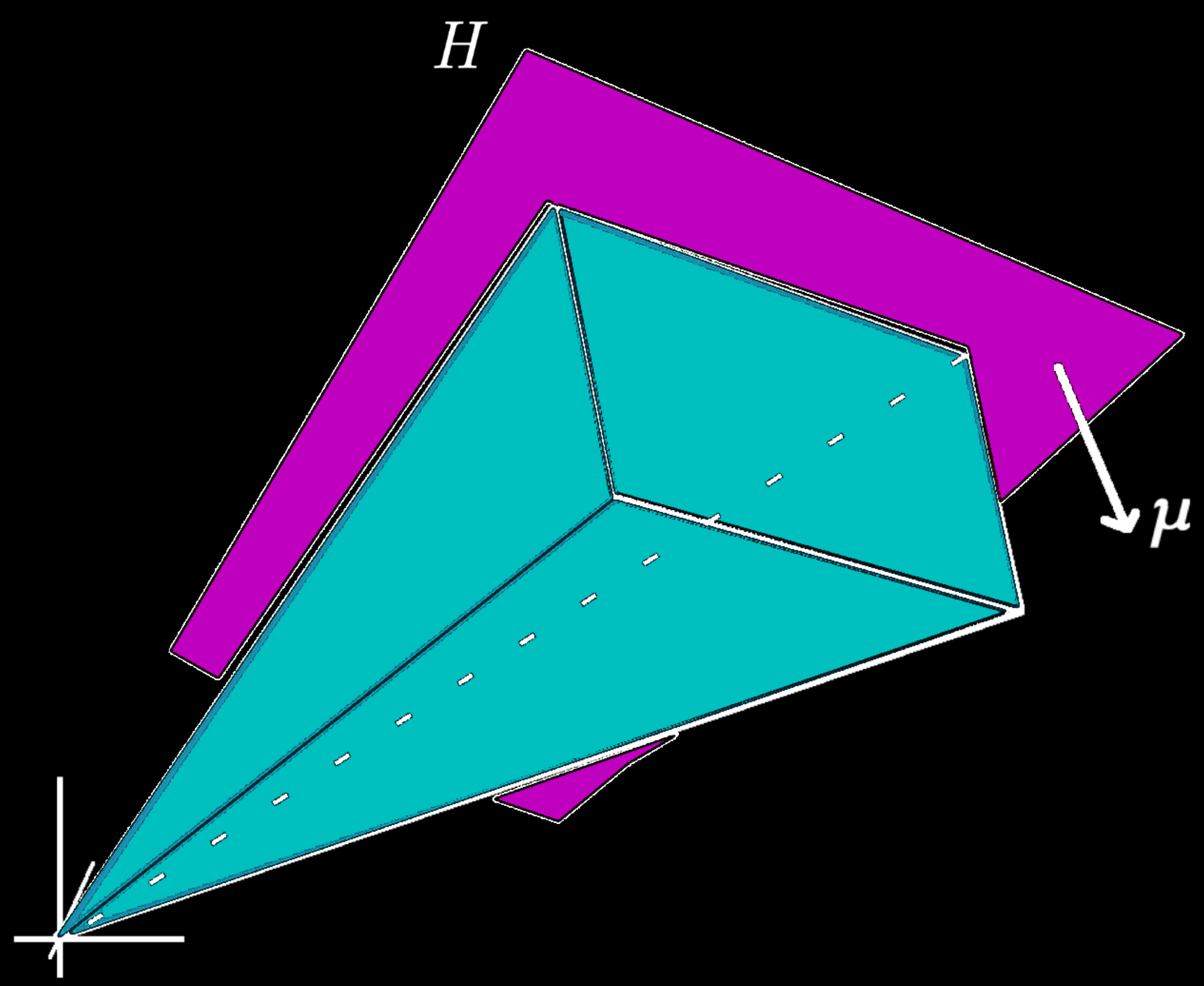

- Corollary:
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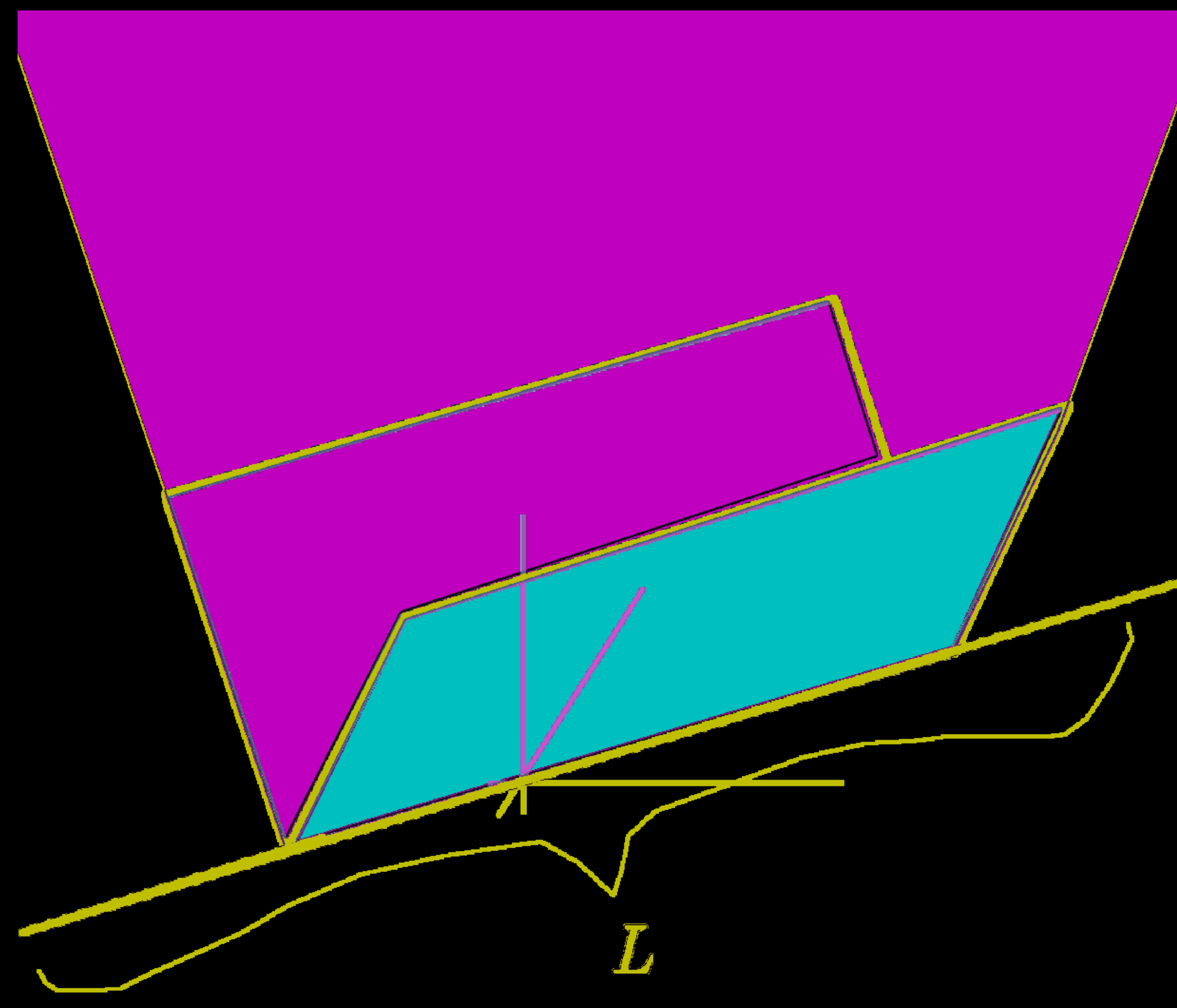
L

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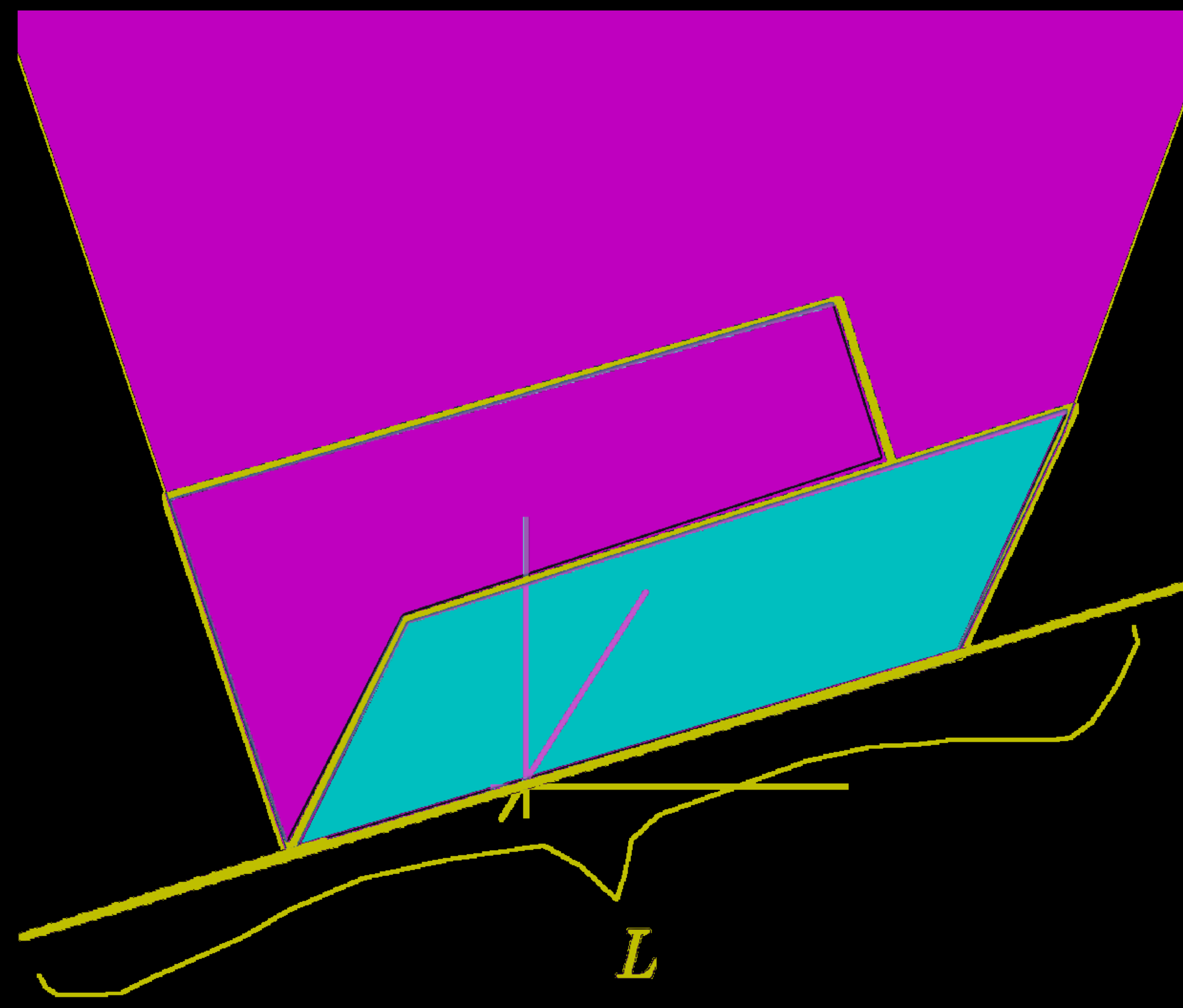

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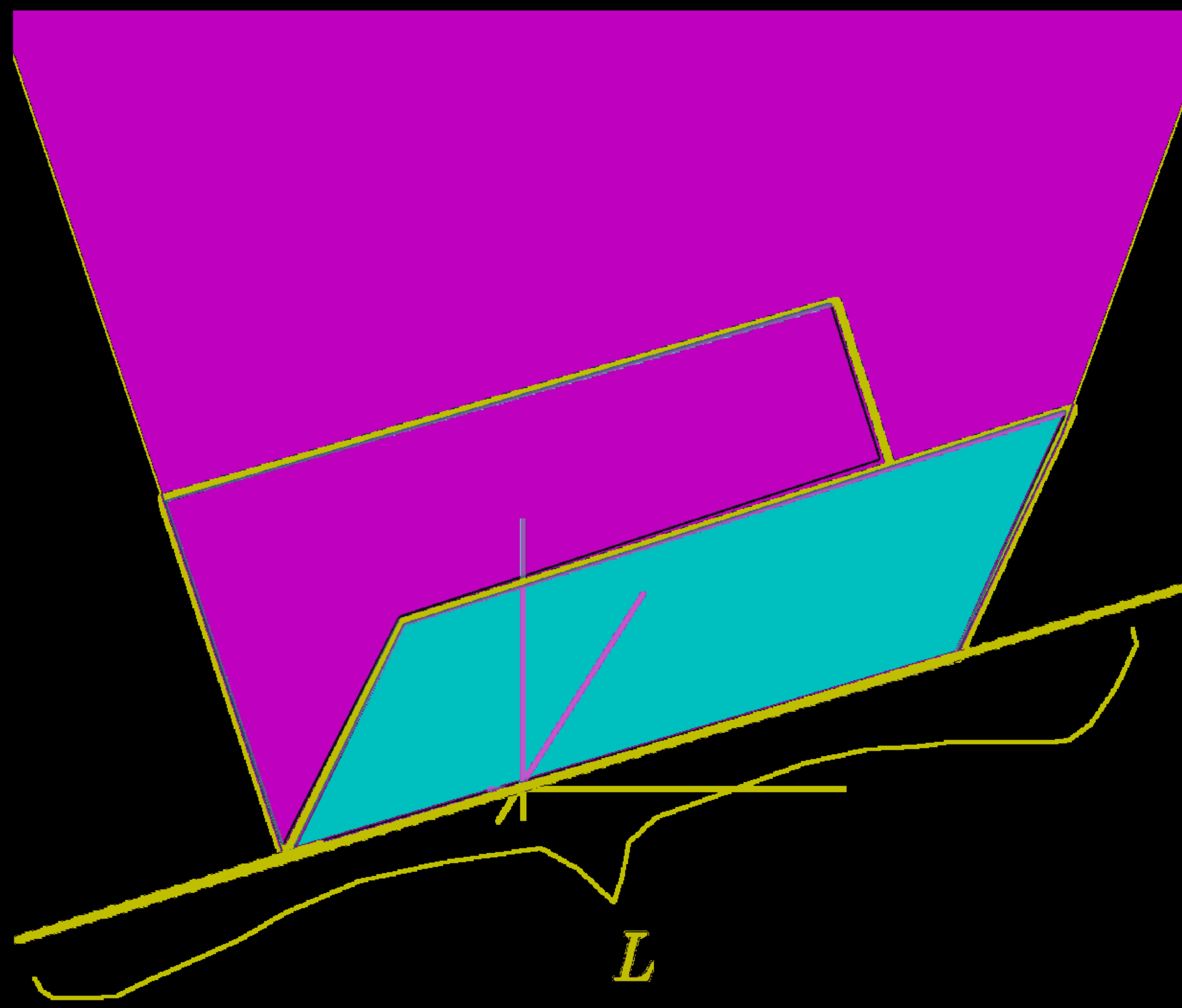

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Moreover, $v_{L(v)}$ can be constructed in polynomial time using *linear programming*.

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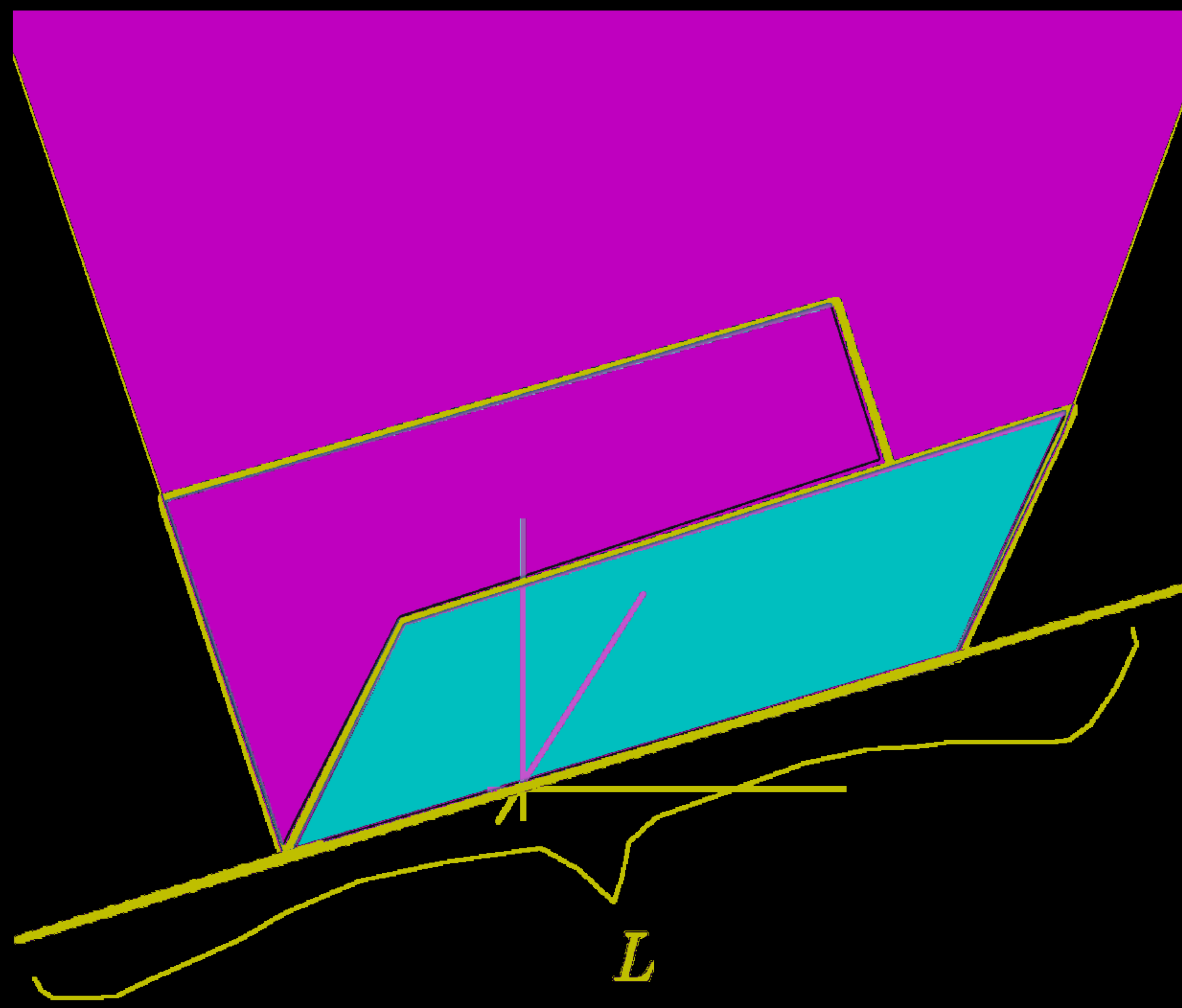

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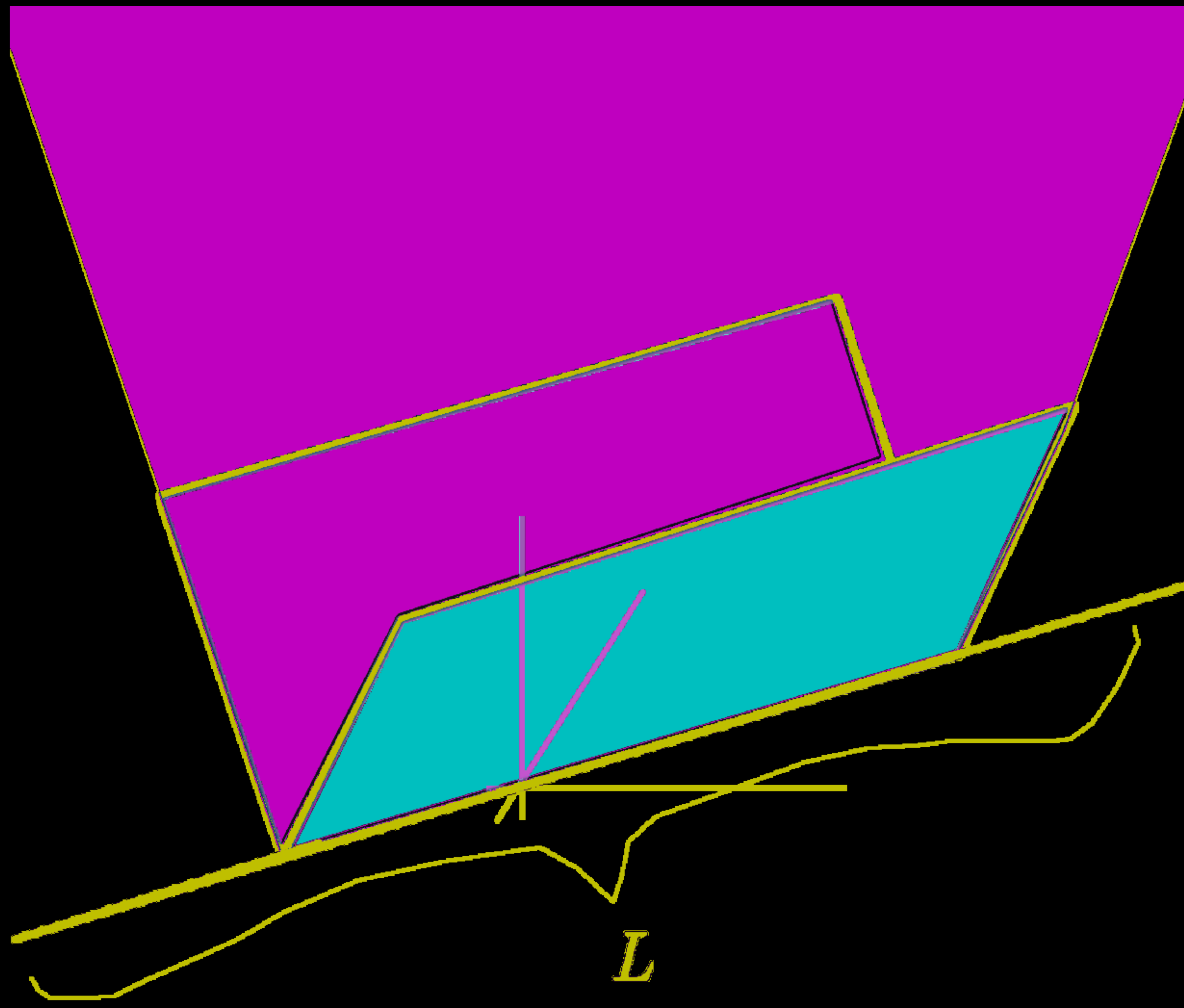
- **Corollary:** There is a poly-time reduction from orbit closure intersection problem to orbit equality problem.



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- If equal, then the orbit closures intersect. If not, the intersection is empty.

Thanks!