Flow polytope volume bounds via polynomial capacity

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Outline

1. Capacity Basics
   - Definition and intuition
   - Gurvits' original application: Computing permanents
   - Coefficient approximation
2. Log-concave Polynomials
   - Various classes of log-concave polynomials
   - Capacity bounds
3. Contingency Tables and Transportation/Flow Polytopes
   - Contingency tables: Definition and generating function
   - Capacity bounds for counting contingency tables
   - Capacity bounds for transportation/flow polytope volume

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Polynomial capacity

Let $\mathbb{R}_+[\boldsymbol{x}]$ denote the set of n-variate polynomials with all coefficients ≥ 0.

Definition: Given $p \in \mathbb{R}_+[\boldsymbol{x}]$ and $\boldsymbol{\alpha} \in \mathbb{R}_+^n$, we define

$$\mathrm{Cap}_{\boldsymbol{\alpha}}(p) := \inf_{\boldsymbol{x}>0} \frac{p(\boldsymbol{x})}{\boldsymbol{x}^{\boldsymbol{\alpha}}} = \inf_{x_1,x_2,\ldots,x_n>0} \frac{p(x_1,x_2,\ldots,x_n)}{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}}.$$

Applications include bounds/approximations for:

- Permanent and mixed discriminant (Gurvits)
- Contingency tables (Barvinok, Barvinok-Hartigan, Gurvits, Brändén-L-Pak)
- Eulerian orientations (Csikvári-Schweitzer)
- Biregular bipartite k-matchings (Gurvits-L)
- Counting/optimization on stable matroids (Straszak-Vishnoi, Anari-Oveis Gharan) and intersection of two general matroids (Anari-Liu-Oveis Gharan-Vinzant)
- Matrix/operator scaling and invariant theory (Allen-Zhu, Bürgisser, Franks, Garg, Gurvits, Li, Oliveira, Reichenbach, Walter, Wigderson)

Intuitions/interpretations of capacity

Definition: Given $p \in \mathbb{R}_+[\boldsymbol{x}]$ and $\boldsymbol{\alpha} \in \mathbb{R}^n_+$, we define

$$\mathrm{Cap}_{\boldsymbol{\alpha}}(p) := \inf_{\boldsymbol{x}>0} \frac{p(\boldsymbol{x})}{\boldsymbol{x}^{\boldsymbol{\alpha}}} = \inf_{x_1,x_2,\ldots,x_n>0} \frac{p(x_1,x_2,\ldots,x_n)}{x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}}.$$

Given $p \in \mathbb{R}_+[\boldsymbol{x}]$ with $p(\mathbf{1}) = 1$, we associate a distribution $\mu \sim p$ supported on $\mathrm{supp}(p) \subset \mathbb{Z}^n_+$ with probability mass function given by:

$$\mu(\boldsymbol{\kappa}) = p_{\boldsymbol{\kappa}} \quad \text{where} \quad p(\boldsymbol{x}) = \sum_{\boldsymbol{\kappa}\in\mathbb{Z}^n_+} p_{\boldsymbol{\kappa}} \boldsymbol{x}^{\boldsymbol{\kappa}}.$$

Intuitions/interpretations of capacity:

1. **Combinatorial:** $\mathrm{Cap}_{\boldsymbol{\alpha}}(p) > 0$ iff $\boldsymbol{\alpha} \in \mathrm{Newt}(p) = \mathrm{hull}(\mathrm{supp}(p))$.
2. **Convexity/optimization:** $\log p(e^{\boldsymbol{y}}) = \log \sum_{\boldsymbol{\kappa}} p_{\boldsymbol{\kappa}} e^{\langle \boldsymbol{y},\boldsymbol{\kappa}\rangle}$ is convex, and $\mathrm{Cap}_{\boldsymbol{\alpha}}(p) = \inf_{\boldsymbol{x}>0} \sum_{\boldsymbol{\kappa}} p_{\boldsymbol{\kappa}} \boldsymbol{x}^{\boldsymbol{\kappa}-\boldsymbol{\alpha}}$ is a geometric program.
3. **Entropic:** Over all distributions ν with $\mathrm{supp}(\nu) = \mathrm{supp}(\mu)$ and $\mathbb{E}[\nu] = \boldsymbol{\alpha}$, the minimum relative entropy $\mathrm{D_{KL}}(\nu \| \mu)$ is $-\log \mathrm{Cap}_{\boldsymbol{\alpha}}(p)$.
4. **Invariant theory:** Scaling problems (positive torus acting on p), null-cone membership, geodesic optimization, etc.

Gurvits' original application: Computing permanents

Given a matrix M with entries in $\mathbb{R}_+$, define the **permanent** of M:

$$\text{per}(M) := \sum_{\sigma \in S_n} \prod_{i=1}^{n} m_{i,\sigma(i)}.$$

Barvinok (I think?): "Like the determinant, but simpler." **Hilarious!**

Why? Exact permanent computation is the canonical #P-hard problem.

Already #P-hard for 0-1 matrices, which is equivalent to counting perfect matchings of a bipartite graph.

Another formulation: Defining $q(\boldsymbol{x}) := \prod_{i=1}^{n} \sum_{j=1}^{n} m_{ij} x_j$, we have

$$\text{per}(M) = \langle x_1 x_2 \cdots x_n \rangle q(\boldsymbol{x}) = \partial_{x_1}\big|_{x_1=0} \cdots \partial_{x_n}\big|_{x_n=0} \, q(\boldsymbol{x}).$$

Crucial capacity concept: q is easy to evaluate, but coefficients are hard to compute. **And:** Sequence of partial derivative operators successively increases hardness from #P-easy to #P-hard.

Capacity and derivatives

Theorem (Gurvits '05)

Given any n-homogeneous, n-variate, "real stable" $p \in \mathbb{R}_+[\boldsymbol{x}]$, we have

$$\mathrm{Cap}_{\mathbf{1}}(\partial_{x_n}|_{x_n=0}\, p) \geq \left(\frac{n-1}{n}\right)^{n-1} \mathrm{Cap}_{\mathbf{1}}(p),$$

where the length of $\mathbf{1}$ *corresponds to the number of remaining variables.*

Corollary: Apply inductively ($\partial_{x_k}|_{x_k=0}$ preserves real stability) to get

$$\mathrm{per}(M) \geq \left(\frac{0}{1}\right)^0 \left(\frac{1}{2}\right)^1 \cdots \left(\frac{n-1}{n}\right)^{n-1} \mathrm{Cap}_{\mathbf{1}}(q) = \frac{n!}{n^n} \cdot \mathrm{Cap}_{\mathbf{1}}(q)$$

where $q(\boldsymbol{x}) = \prod_{i=1}^n \sum_{j=1}^n m_{ij} x_j$.

When M doubly stochastic (row sums = col sums = 1):
$\mathrm{Cap}_{\mathbf{1}}(q) = 1 \implies$ van der Waerden bound (Egorychev, Falikman ∼'80).

This gives a **closed-form bound** via some analysis of a convex program.

Coefficient approximation

Theorem (Gurvits '08)

Given any d-homogeneous, n-variate, "strongly log-concave" (SLC) $p \in \mathbb{R}_+[\boldsymbol{x}]$ and any $\boldsymbol{\kappa} \in \operatorname{supp}(p)$, we have

$$\langle \boldsymbol{x}^{\boldsymbol{\kappa}} \rangle p(\boldsymbol{x}) \geq \binom{d}{\boldsymbol{\kappa}} \frac{\kappa_1^{\kappa_1} \cdots \kappa_n^{\kappa_n}}{d^d} \operatorname{Cap}_{\boldsymbol{\kappa}}(p).$$

Other direction: $\langle \boldsymbol{x}^{\boldsymbol{\kappa}} \rangle p(\boldsymbol{x}) \leq \inf_{\boldsymbol{x}>0} \frac{p(\boldsymbol{x})}{\boldsymbol{x}^{\boldsymbol{\kappa}}} = \operatorname{Cap}_{\boldsymbol{\kappa}}(p)$. **For example:**

$$\operatorname{Cap}_{\mathbf{1}}(q) \geq \langle x_1 x_2 \cdots x_n \rangle q(\boldsymbol{x}) = \operatorname{per}(M) \geq \frac{n!}{n^n} \cdot \operatorname{Cap}_{\mathbf{1}}(q) \geq e^{-n} \cdot \operatorname{Cap}_{\mathbf{1}}(q).$$

Upshot: Coefficient approximation via convex programming.

Now: Coefficients of generating functions count combinatorial objects.

Corollary: If "SLC" generating function, we can approximately count.

What is real stable / SLC? What other classes of polynomials?

Outline

Real stable polynomials

Definition: $p \in \mathbb{R}[\boldsymbol{x}]$ is **real stable** if

$$z_1, \ldots, z_n \in \mathcal{H}_+ \text{ (upper half-plane of } \mathbb{C}) \quad \Longrightarrow \quad p(z_1, \ldots, z_n) \neq 0.$$

Univariate case: Equivalent to real-rooted due to conjugate pairs.

Classic examples: Elementary symmetric polynomials, $\det(\sum_i x_i A_i)$ for PSD A_i, product of linear forms with non-negative entries, etc.

Newton's inequalities: If $p(t) = \sum_{k=0}^{d} p_k t^k$ is real-rooted, then

$$\left(\frac{p_k}{\binom{d}{k}}\right)^2 \geq \left(\frac{p_{k-1}}{\binom{d}{k-1}}\right)\left(\frac{p_{k+1}}{\binom{d}{k+1}}\right).$$

Another name is **ultra log-concave** coefficients (implies log-concavity).

Generalizes to real stable $p \in \mathbb{R}[\boldsymbol{x}]$ via the **strong Rayleigh inequalities**.

Upshot: Real stability is a certain strong form of log-concavity.

Next question: Is real-rooted equivalent to Newton's inequalities? **No...**

SLC / CLC / Lorentzian polynomials

Definition: d-homogeneous, n-variate $p \in \mathbb{R}_+[\boldsymbol{x}]$ is **strongly log-concave** (Gurvits) / **completely log-concave** (Anari-Liu-Ovies Gharan-Vinzant) / **Lorentzian** (Brändén-Huh) if one of the following equiv. conditions holds:

1. $\nabla_{\boldsymbol{v}_1} \cdots \nabla_{\boldsymbol{v}_k} p$ is log-concave in $\mathbb{R}^n_+$ for all $k \geq 0$ and $\boldsymbol{v}_i \in \mathbb{R}^n_+$.
2. $\partial_{x_{k_1}} \cdots \partial_{x_{k_{d-2}}} p$ is a quadratic form with signature $(+, -, \cdots, -)$ for all choices $k_i \in [n]$, and the support of p is "matroidal".
3. $(\nabla_{\boldsymbol{v}_1} \cdots \nabla_{\boldsymbol{v}_d} p)^2 \geq (\nabla_{\boldsymbol{v}_1} \nabla_{\boldsymbol{v}_1} \nabla_{\boldsymbol{v}_3} \cdots \nabla_{\boldsymbol{v}_d} p) \cdot (\nabla_{\boldsymbol{v}_2} \nabla_{\boldsymbol{v}_2} \nabla_{\boldsymbol{v}_3} \cdots \nabla_{\boldsymbol{v}_d} p)$ for all choices of $\boldsymbol{v}_i \in \mathbb{R}^n_+$. (Alexandrov-Fenchel)

Hot topic: Hodge theory for matroids, resolution of Mason's conjectures, approximating intersection of two matroids, etc.

Examples: $\text{vol}(\sum_i x_i K_i)$ for compact convex K_i, matroid basis generating polynomials, normalized Schur polynomials [HMMD '19], more?

Bivariate homogeneous case: $p(t, s) = \sum_{k=0}^{d} p_k t^k s^{d-k}$. Condition (2) implies p is SLC iff coefficients are ultra log-concave (with no "gaps").

Denormalized Lorentzian polynomials

Definition: d-homogeneous, n-variate $p \in \mathbb{R}_+[\boldsymbol{x}]$ is **denormalized Lorentzian** (DL) if its normalization is Lorentzian:

$$N[p] := \sum_{\boldsymbol{\kappa} \in \mathrm{supp}(p)} \binom{d}{\boldsymbol{\kappa}} p_{\boldsymbol{\kappa}} \boldsymbol{x}^{\boldsymbol{\kappa}}.$$

Examples: Schur polynomials, contingency tables generating polynomials, bivariate homogeneous polynomials with log-concave coefficients, more?

Log-concave polynomials:

1. Real stability generalizes real-rootedness, which is stronger than ultra log-concave coefficients (Newton's inequalities).
2. Lorentzian generalizes real stable to give the "correct" generalization of ultra log-concavity. (**Downside:** No root location condition.)
3. Denormalized Lorentzian polynomials piggyback off Lorentzian to give the "correct" generalization of log-concave coefficients.

Super bonus: All classes closed under taking products of polynomials.

Capacity and denormalized Lorentzian polynomials

Before: Coefficient bounds for SLC (and real stable) polynomials.

Theorem (Brändén-L-Pak '20)

Given any d-homogeneous, n-variate, denormalized Lorentzian $p \in \mathbb{R}_+[\boldsymbol{x}]$ and any $\boldsymbol{\kappa} \in \operatorname{supp}(p)$, we have

$$\langle \boldsymbol{x}^{\boldsymbol{\kappa}} \rangle p(\boldsymbol{x}) \geq \left[\prod_{i=2}^{n} \frac{\kappa_i^{\kappa_i}}{(\kappa_i + 1)^{\kappa_i + 1}} \right] \operatorname{Cap}_{\boldsymbol{\kappa}}(p).$$

A few things to note:

- Starting at $i = 2$ is **not** a typo.
- No dependence on the degree d, but a stronger/more symmetric version of the theorem depends on per-variable degree.
- Approximate coefficients of denormalized Lorentzian polynomials.

Corollary: $\langle \boldsymbol{x}^{\boldsymbol{\kappa}} \rangle p(\boldsymbol{x}) \geq e^{-(n-1)} \left[\prod_{i=2}^{n} \frac{1}{\kappa_i + 1} \right] \operatorname{Cap}_{\boldsymbol{\kappa}}(p).$

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Contingency tables

Definition: Given $\boldsymbol{\alpha} \in \mathbb{N}^m$ and $\boldsymbol{\beta} \in \mathbb{N}^n$, a **contingency table** is an $m \times n$ matrix of non-negative integers such that the row sums and column sums are $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ respectively ($\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ called the **marginals** of M).

Examples: The permutation matrices are the contingency tables with $\boldsymbol{\alpha} = \boldsymbol{\beta} = \mathbf{1}$. The d-regular bipartite multigraphs are the contingency tables with $\boldsymbol{\alpha} = \boldsymbol{\beta} = d \cdot \mathbf{1}$. Contingency tables with $\boldsymbol{\alpha} = (1,4)$ and $\boldsymbol{\beta} = (1,2,2)$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

Generating function: Fix matrix M, and to entry m_{ij} associate $(x_i y_j)^{m_{ij}}$. M has marginals $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ iff $\prod_{i=1}^m \prod_{j=1}^n (x_i y_j)^{m_{ij}} = \boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{y}^{\boldsymbol{\beta}}$. **Therefore:**

$$g(\boldsymbol{x}, \boldsymbol{y}) := \prod_{i=1}^m \prod_{j=1}^n \frac{1}{1 - x_i y_j} = \prod_{i=1}^m \prod_{j=1}^n \sum_{k=0}^\infty (x_i y_j)^k = \sum_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \cdot \boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{y}^{\boldsymbol{\beta}},$$

where $\mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ counts contingency tables with the given marginals.

Capacity bounds for contingency tables

Goal: Apply capacity bounds to generating function.

Problems: Not a polynomial, not homogeneous. **We can fix it though:**

$$\prod_{i=1}^{m}\prod_{j=1}^{n}\sum_{k=0}^{\infty}(x_i y_j)^k \;\to\; \prod_{i=1}^{m}\prod_{j=1}^{n}\sum_{k=0}^{K} x_i^k y_j^{K-k} = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}} \mathrm{CT}_K(\boldsymbol{\alpha},\boldsymbol{\beta}) \cdot \boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{y}^{mK\cdot\mathbf{1}-\boldsymbol{\beta}},$$

where $\mathrm{CT}_K(\boldsymbol{\alpha},\boldsymbol{\beta})$ is the number of tables with entries bounded by K.

Now: Generating function is a product of $x_i^K + x_i^{K-1}y_j + \cdots + y_j^K$, which is a bivariate homogeneous polynomial with log-concave coefficients.

Therefore: The new generating function is denormalized Lorentzian.

Pick large K and apply the capacity bound (omitting some details) to get:

$$\mathrm{CT}_K(\boldsymbol{\alpha},\boldsymbol{\beta}) \geq e^{-(m+n-1)} \left[\prod_{i=2}^{m} \frac{1}{\alpha_i+1} \prod_{j=1}^{n} \frac{1}{\beta_j+1}\right] \mathrm{Cap}_{(\boldsymbol{\alpha}, mK\cdot\mathbf{1}-\boldsymbol{\beta})}(\tilde{g}_K),$$

where $\tilde{g}_K$ is the above twisted, truncated generating function.

Capacity bounds for the original generating function

Last slide: For K large, we have

$$\mathrm{CT}_K(\boldsymbol{\alpha},\boldsymbol{\beta}) \geq e^{-(m+n-1)} \left[\prod_{i=2}^{m} \frac{1}{\alpha_i+1} \prod_{j=1}^{n} \frac{1}{\beta_j+1}\right] \mathrm{Cap}_{(\boldsymbol{\alpha}, mK\cdot\mathbf{1}-\boldsymbol{\beta})}(\tilde{g}_K).$$

Notice: $\mathrm{Cap}_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(q) = \inf_{\boldsymbol{x},\boldsymbol{y}>0} \dfrac{q(\boldsymbol{x},\boldsymbol{y})}{\boldsymbol{x}^{\boldsymbol{\alpha}}\boldsymbol{y}^{\boldsymbol{\beta}}} = \inf_{\boldsymbol{x},\boldsymbol{y}>0} \dfrac{\boldsymbol{y}^{L\cdot\mathbf{1}} \cdot q(\boldsymbol{x},\boldsymbol{y}^{-\mathbf{1}})}{\boldsymbol{x}^{\boldsymbol{\alpha}}\boldsymbol{y}^{L\cdot\mathbf{1}-\boldsymbol{\beta}}}.$

Now: The constant without the capacity factor is independent of K, and $\mathrm{CT}(\boldsymbol{\alpha},\boldsymbol{\beta}) = \mathrm{CT}_K(\boldsymbol{\alpha},\boldsymbol{\beta})$ for large enough K. **So twist back and limit:**

$$\mathrm{CT}(\boldsymbol{\alpha},\boldsymbol{\beta}) \geq e^{-(m+n-1)} \left[\prod_{i=2}^{m} \frac{1}{\alpha_i+1} \prod_{j=1}^{n} \frac{1}{\beta_j+1}\right] \mathrm{Cap}_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(g).$$

New problem: What does $\mathrm{Cap}_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(g)$ mean? **Answer:**

$$\mathrm{Cap}_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(g) := \inf_{\boldsymbol{x}\in(0,1)^m, \boldsymbol{y}\in(0,1)^n} \frac{\prod_{i=1}^{m}\prod_{j=1}^{n}(1-x_i y_j)^{-1}}{\boldsymbol{x}^{\boldsymbol{\alpha}}\boldsymbol{y}^{\boldsymbol{\beta}}}.$$

Transportation/flow polytopes

Definition: Given $\boldsymbol{\alpha} \in \mathbb{N}^m$ and $\boldsymbol{\beta} \in \mathbb{N}^n$, the associated **transportation polytope** is the set of all $m \times n$ matrices with $\mathbb{R}_+$ entries such that the row sums and column sums are $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ respectively. Given K, the associated **flow poytope** has the extra constraint that all entries are bounded by K.

That is: Contingency tables are the integer points of these polytopes.

Therefore: We can extract volume from $\mathrm{CT}(M\boldsymbol{\alpha}, M\boldsymbol{\beta})$ as $M \to \infty$.

How? E.g., the Ehrhart polynomial of an integral polytope counts its integer points, and its leading coefficient is the volume of the polytope.

Now: Since the dimension of the transportation polytope $\mathcal{T}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is $(m-1)(n-1)$, we want to bound

$$\mathrm{vol}(\mathcal{T}(\boldsymbol{\alpha}, \boldsymbol{\beta})) = \lim_{M \to \infty} \frac{\mathrm{CT}(M\boldsymbol{\alpha}, M\boldsymbol{\beta})}{M^{(m-1)(n-1)}}$$

via our capacity bound on $\mathrm{CT}(M\boldsymbol{\alpha}, M\boldsymbol{\beta})$.

Volume bounds via capacity

Last slide: $\mathrm{vol}(\mathcal{T}(\boldsymbol{\alpha},\boldsymbol{\beta})) = \lim_{M\to\infty} \frac{\mathrm{CT}(M\boldsymbol{\alpha}, M\boldsymbol{\beta})}{M^{(m-1)(n-1)}}$.

From before, we have our capacity bound:

$$\begin{aligned}\mathrm{CT}(M\boldsymbol{\alpha}, M\boldsymbol{\beta}) &\geq e^{-(m+n-1)} \left[\prod_{i=2}^{m} \frac{1}{M\alpha_i + 1} \prod_{j=1}^{n} \frac{1}{M\beta_j + 1}\right] \mathrm{Cap}_{(M\boldsymbol{\alpha}, M\boldsymbol{\beta})}(g) \\ &= (eM)^{-(m+n-1)} \left[\prod_{i=2}^{m} \frac{1}{\alpha_i + \frac{1}{M}} \prod_{j=1}^{n} \frac{1}{\beta_j + \frac{1}{M}}\right] \mathrm{Cap}_{(M\boldsymbol{\alpha}, M\boldsymbol{\beta})}(g).\end{aligned}$$

Now add in the limit:

$$\begin{aligned}\lim_{M\to\infty} \frac{\mathrm{CT}(M\boldsymbol{\alpha}, M\boldsymbol{\beta})}{M^{mn-(m+n-1)}} &= e^{-(m+n-1)} \prod_{i=2}^{m} \frac{1}{\alpha_i} \prod_{j=1}^{n} \frac{1}{\beta_j} \\ &\times \lim_{M\to\infty} \left[\inf_{\boldsymbol{x}\in(0,1)^m, \boldsymbol{y}\in(0,1)^n} \frac{\prod_{i=1}^m \prod_{j=1}^n (1 - x_i y_j)^{-1}}{M^{mn} \cdot \boldsymbol{x}^{M\boldsymbol{\alpha}} \boldsymbol{y}^{M\boldsymbol{\beta}}}\right].\end{aligned}$$

Volume bounds via capacity

Last piece: $\lim_{M\to\infty}\left[\inf_{0<\boldsymbol{x},\boldsymbol{y}<1}\frac{\prod_{i=1}^m\prod_{j=1}^n(1-x_iy_j)^{-1}}{M^{mn}\cdot\boldsymbol{x}^{M\boldsymbol{\alpha}}\boldsymbol{y}^{M\boldsymbol{\beta}}}\right].$

Omitting details, this limit actually has a nice form:

$$\inf_{0<\boldsymbol{x},\boldsymbol{y}<1}\frac{\prod_{i=1}^m\prod_{j=1}^n(-\log(x_iy_j))^{-1}}{\boldsymbol{x}^{\boldsymbol{\alpha}}\boldsymbol{y}^{\boldsymbol{\beta}}}=\mathrm{Cap}_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\left(\prod_{i=1}^m\prod_{j=1}^n\frac{-1}{\log(x_iy_j)}\right).$$

Further, when $\boldsymbol{\alpha}=\alpha_0\cdot\mathbf{1}$ and $\boldsymbol{\beta}=\beta_0\cdot\mathbf{1}$, we have the **closed-form** bound:

$$\mathrm{vol}(\mathcal{T}_{\alpha_0\cdot\mathbf{1},\beta_0\cdot\mathbf{1}})\geq\frac{(e\cdot m\alpha_0)^{(m-1)(n-1)}}{m^{m(n-1)+1}n^{n(m-1)}}.$$

For the **Birkhoff polytope** with $\alpha_0=\beta_0=1$ and $m=n$:

$$\mathrm{vol}(\mathcal{T}_{\mathbf{1},\mathbf{1}})\geq\frac{(en)^{(n-1)^2}}{n^{2n^2-2n+1}}=\frac{e^{(n-1)^2}}{n^{n^2}}=e^{-n^2\log n+n^2-2n+1}.$$

First two terms coincide with the true asymptotics [Canfield-McKay '07].