Anticoncentration and central limit theorems from zero-free regions

Introductory talk

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Goal of this (30 minute) talk

Explain the three nouns in the title of the talk:

- Anticoncentration (and concentration)
- Central Limit Theorems
- Zero-free regions (as related to this talk)

Large random systems often exhibit predictable behavior

Let's review a prototypical example. For $j = 0, 1, 2, \ldots$ let X_j be independent and identically distributed. Set $\mathbb{E}X_j = \mu$ and $\mathrm{Var}(X_j) = \sigma^2$.

For each n set $S_n = X_1 + \ldots + X_n = \sum_{j=1}^{n} X_j$.

Then $\mathbb{E}S_n = n\mu$ and $\mathrm{Var}(S_n) = n\sigma^2$.

$$S_n \approx n\mu + \sqrt{n}\sigma Z$$

where Z is a standard (mean zero, variance one) Gaussian random variable.

The $n\mu$ term is the **Law of Large Numbers** (LLN) and the $\sqrt{n}\sigma Z$ term is the **Central Limit Theorem** (CLT).

What does the law of large numbers say?

X_j independent, identically distributed with $\mathbb{E}X_j = \mu, \mathrm{Var}(X_j) = \sigma^2$; $S_n = X_1 + \cdots + X_n$.

$$S_n \approx n\mu + \sqrt{n}\sigma Z\,.$$

$$\frac{S_n}{n} \approx \mu + (\text{something small})\,.$$

S_n is **concentrated** around $n\mu$. It is unlikely to differ from $n\mu$ by too much.

Concentration is when a random variable is unlikely to differ from something predictable by too much.

What does the central limit theorem say?

X_j independent, identically distributed with $\mathbb{E}X_j = \mu, \mathrm{Var}(X_j) = \sigma^2$; $S_n = X_1 + \cdots + X_n$.

$$\text{If we set} \quad \hat{S}_n = \frac{S_n - n\mu}{\sqrt{n}\sigma} \quad \text{then} \quad \mathbb{P}(\hat{S}_n \geq t) \approx \mathbb{P}(Z \geq t)\,,$$

where Z is a standard Gaussian random variable.

On the one hand this says that $\hat{S}_n$ is unlikely to be big

$$\mathbb{P}(|\hat{S}_n| \geq t) \approx \mathbb{P}(|Z| \geq t) \leq e^{-t^2/4}$$

for t big enough (and $n \to \infty$).

On the other hand, it says that $|\hat{S}_n|$ isn't just the random variable that is always 0:

$$\mathbb{P}(|\hat{S}_n| \geq t) \approx \mathbb{P}(|Z| \geq t) \geq e^{-t^2}\,.$$

What does the central limit theorem say? (Summary)

X_j independent, identically distributed with $\mathbb{E}X_j = \mu, \mathrm{Var}(X_j) = \sigma^2$;
$S_n = X_1 + \cdots + X_n$.

$$\text{If we set} \quad \hat{S}_n = \frac{S_n - n\mu}{\sqrt{n}\sigma} \quad \text{then} \quad \mathbb{P}(\hat{S}_n \geq t) \approx \mathbb{P}(Z \geq t)\,,$$

where Z is a standard Gaussian random variable.

The central limit theorem gives both:

- **Concentration**: $\hat{S}_n$ isn't likely to be too big.
- **Anticoncentration**: $\hat{S}_n$ is *non-degenerate*, it isn't always 0.

The story thus far

Given a random variable Y (think of Y as like our random sum S_n):

- **Concentration** is a statement saying that Y is likely to exhibit deterministic behavior (e.g., Y is close to a constant).
- **Anticoncentration** is a statement showing that Y is *non-degenerate* (e.g., Y is looks different than a constant).
- A **Central Limit Theorem** is when $\dfrac{Y - \mathbb{E}Y}{\sqrt{\mathrm{Var}(Y)}}$ looks like a Gaussian random variable.

Lots of machinery for concentration and central limit theorems in certain cases

Recall the sum $S_n = X_1 + \ldots + X_n$, where $X_1, \ldots, X_n$ are independent. Then we get concentration and a central limit theorem from classical results.

Morally, if we have a nice function F and $X_1, \ldots, X_n$ are independent and not too large, then concentration of $F(X_1, \ldots, X_n)$ can be shown. This is an entire (sub)field of probability.

Similarly, if F doesn't depend too much on any given coordinate, then $F(X_1, \ldots, X_n)$ satisfies a central limit theorem.

Anticoncentration without a central limit theorem

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed with $\mathbb{P}(X_j = 1) = \mathbb{P}(X_j = -1) = 1/2$. Let $a_1, \ldots, a_n$ be real numbers with $|a_j| \geq 1$. Set

$$S = a_1 X_1 + a_2 X_2 + \cdots + a_n X_n \,.$$

The random variable S might look very far from a Gaussian (i.e. no CLT holds) if we choose a_j in an adversarial way (e.g. $a_j = 3^j$). Can we deduce uniform anticoncentration?

Theorem (Erdős, 1945)

$$\max_t \mathbb{P}(|S - t| < 1/2) \leq \frac{1}{\sqrt{n}}$$

For any given interval of length 1, S is unlikely to fall in that interval.

A sidenote: Anticoncentration is less understood than concentration

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed with $\mathbb{P}(X_j = 1) = \mathbb{P}(X_j = -1) = 1/2$. Set $X = (X_1, \ldots, X_n)^T$ and let M be an $n \times n$ matrix with each entry having $|M_{i,j}| \geq 1$.

Conjecture (Quadratic Littlewood-Offord Problem)

$$\mathbb{P}(X^T M X = 0) \leq \frac{C}{\sqrt{n}}$$

This is known up to log factors (see: Costello, Costello-Tao-Vu, Kwan-Sauermann, Meka-Nguyen-Vu)

These all use independence

Every theorem I've told you so far is about a function of a random environment: given $X_1, \dots, X_n$ independent and a function F, I've told you about concentration, anticoncentration, and central limit theorems for $F(X_1, \dots, X_n)$ under certain assumptions on X_j's and F.

Guiding Question

How can we expand this story without using ***any*** *independence assumption?*

Towards our third noun in the title: "zero free regions"...

Probability generating functions

Let $X \in \{0, 1, \ldots, n\}$ be a random variable. Define the *probability generating function* f_X by

$$f_X(z) = \mathbb{E}z^X = \sum_k \mathbb{P}(X = k)z^k\,.$$

Properties:

- f_X is a polynomial of degree n.
- The coefficients of f_X are non-negative.
- $f_X(1) = 1$.

These three properties characterize probability generating functions.

Some more properties of probability generating functions

If X and Y are independent, then:

$$f_{X+Y}(z) = \mathbb{E}z^{X+Y} = \mathbb{E}z^X \mathbb{E}z^Y = f_X(z) f_Y(z)\,.$$

Theorem (Harper, 1967)

Let $X \in \{0, 1, \ldots, n\}$ be a random variable and suppose that all roots of f_X are real. Then $\hat{X} = \dfrac{X - \mathbb{E}X}{\sqrt{\mathrm{Var}(X)}}$ is close to a standard Gaussian if and only if $\mathrm{Var}(X)$ is large.

Proving the CLT when real-rooted

Assume f_X has only real roots, let $\{\zeta_j\}$ be these roots. Note that they can't be positive since f_X has non-negative coefficients. Factor

$$f_X(z) = \prod_{j=1}^{n} \left(\frac{z}{1-\zeta_j} + \frac{-\zeta_i}{1-\zeta_j} \right).$$

$$Y_j := \begin{cases} 1 & \text{w/ prob. } \frac{1}{1-\zeta_j} \\ 0 & \text{w/ prob. } \frac{-\zeta_j}{1-\zeta_j} \end{cases}$$

$$X = Y_1 + \cdots + Y_n,$$

The ordinary CLT says: X has a CLT provided $\mathrm{Var}(X)$ is large.

CLT when real-rooted

Theorem (Harper, 1967)

Let $X \in \{0, 1, \ldots, n\}$ be a random variable and suppose that all roots of f_X are real. Then $\hat{X} = \dfrac{X - \mathbb{E}X}{\sqrt{\mathrm{Var}(X)}}$ is close to a standard Gaussian if and only if $\mathrm{Var}(X)$ is large.

Proof.

This is really just the ordinary CLT in disguise. □

A quantitative central limit theorem also provides anticoncentration in this case:

$$\max_s \mathbb{P}(X = s) \leq \frac{C}{\sqrt{\mathrm{Var}(X)}}\,.$$

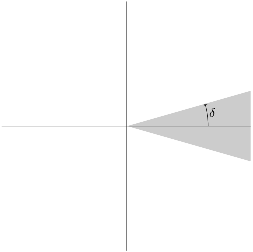
Beyond real roots, and more context on zero-freeness

Goals for the next talk:

1. How far can we push this? We will see (sharp, quantitative) central limit theorems assuming zero-freeness in either a neighborhood of 1 or a sector/cone.
2. What are some cases in which we have zero-freeness? What does this assumption "mean"? There is a deep connection between zero-freeness and absence of phase transitions in physics (so-called Lee-Yang theory).
3. How does anti-concentration fit into this picture?

A taste of these results

Let $X \in \{0, 1, \ldots, n\}$ be a random variable, $f_X(z) = \mathbb{E}z^X$. Assume f_X is zero-free in the sector $\{|\arg(z)| \leq \delta\}$, i.e. no zeros in this gray region:



Theorem (M.-Sahasrabudhe)

In this context, $\hat{X} = \dfrac{X - \mathbb{E}X}{\sqrt{\mathrm{Var}(X)}}$ *is close to a standard Gaussian provided* $\delta^2 \mathrm{Var}(X)$ *is large.*

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Probability generating function

Let $X \in \{0, 1, \ldots, n\}$ be a random variable. Its *probability generating function* is

$$f_X(z) = \mathbb{E}z^X = \sum_{k=0}^{n} \mathbb{P}(X = k) z^k \,.$$

$f_X(z)$ is a polynomial with *non-negative coefficients* and $f_X(1) = 1$.

Note the characteristic function $\mathbb{E}e^{itX} = f(e^{it})$. Will typically write $\mu = \mathbb{E}X$ and $\sigma^2 = \mathrm{Var}X$.

Guiding Question

What is the relationship between the roots of f_X and the behavior of the random variable X?

A connection to statistical mechanics

(Meta)theorem (Yang-Lee '52)

There is a connection between phase transitions and zero-free regions. A way to prove a lack of a phase transition is to prove a zero-free region.

Two now-classical examples:

- The Ising model (Lee-Yang, '52).
- The "monomer dimer model" a.k.a. random matchings from a graph (Heilmann-Lieb, '72).

Vague idea of Lee-Yang theory

Given a finite graph G, and parameters $\gamma \in (0,1)$ and z, the partition function of the Ising model is the polynomial

$$Z_{G,\gamma}(z) = \sum_{S \subset G} \gamma^{|\partial S|} z^{|S|}$$

where S is a sum over all subsets of vertices in G and $|\partial S|$ is the number of edges in G with one vertex in S and one not. The form is not important for now, but it is a polynomial.

On a graph like $\mathbb{Z}^d$, we can take a subsequence of graphs $G_n \uparrow \mathbb{Z}^d$.

A thermodynamic quantity known as the *free energy* is defined as

$$\Phi(z) = \lim_{n \to \infty} \frac{1}{|G_n|} \log Z_{G_n,\gamma}(z)\,.$$

Abstracting it a bit

We have a sequence of polynomials P_n of degree n and the *free energy* is the limiting function

$$\Phi(z) = \lim_{n\to\infty} \frac{1}{n} \log P_n(z)\,.$$

Yang-Lee (meta)Theorem/definition: a phase transition is a point $z \geq 0$ at which Φ fails to be analytic.

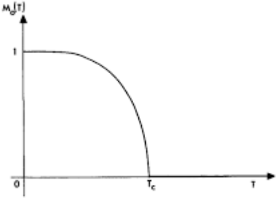


Fig. 1.3. The spontaneous magnetization M_0 as a function of temperature.

Lee-Yang continued

We have a sequence of polynomials P_n of degree n and the *free energy* is the limiting function

$$\Phi(z) = \lim_{n\to\infty} \frac{1}{n} \log P_n(z)\,.$$

Lee-Yang: no phase transition for $z \in [a, b] \subset \mathbb{R}$ means that Φ is analytic in a neighborhood of $[a, b]$.

P_n is a polynomial so it is analytic, so $\log P_n$ is analytic provided P_n is zero-free.

Theorem (Lee-Yang, '52)

On ***any graph****, the partition function of the Ising model has all roots on the unit circle. In particular, all roots of P_n have $|z| = 1$.*

Theorem (Yang-Lee, '52)

As z varies, the only place the Ising model may have a phase transition is at $z = 1$.

Another example: Matchings in a graph

Given a finite graph G, a *matching* M is a collection of edges that do not touch each other.

Define

$$F_G(z) = \sum_M z^{|M|}$$

where the sum is over all matchings.

Theorem (Heilmann-Lieb, '72)

For any graph G, all roots of $F_G(z)$ are real (and thus non-positive).

Note that if I fix $\lambda > 0$ then $F_G(\lambda z)/F_G(\lambda)$ is the generating function for the number of edges in a random matching when I choose a matching with probability proportional to $\lambda^{|M|}$. This is real rooted, so we get a CLT for $|M|$ if and only if $\mathrm{Var}(|M|) \to \infty$. (see Godsil, and/or Kahn)

Zero-free in a sector

Philosophy: Random variables for which f_X has no roots in $\{z : |\arg(z)| < \delta\}$ behave like functions of roughly independent stuff. In particular, they should behave roughly like i.i.d. sums, up to some "distortion" depending on δ.

So we should have:

- CLT
- Linear variance
- Anti-concentration

CLT when zero-free in a sector

Theorem (M.-Sahasrabudhe)

Let $X \in \{0, \dots, n\}$ suppose all zeros ζ of $f_X(z)$ $|\arg(\zeta)| \geq \delta$. Then

$$\sup_t \left| \mathbb{P}\left(\frac{X - \mathbb{E}X}{\sqrt{\mathrm{Var}(X)}} \leq t \right) - \mathbb{P}(Z \leq t) \right| \leq \frac{C}{\delta \sqrt{\mathrm{Var}(X)}}$$

where Z is a standard Gaussian.

Like sums of independent variables, CLT if and only if $\mathrm{Var}(X) \to \infty$ (for when δ is fixed).

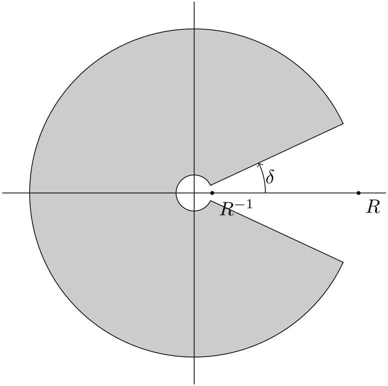
This is tight up to the constant C.

Anti-concentration theorem

Theorem (M.-Sahasrabudhe)

Let $X \in \{0, \dots, n\}$ with $\mathbb{P}(X = 0)\mathbb{P}(X = n) > 0$ and suppose all zeros ζ of $f_X(z)$ satisfy $R^{-1} \le |\zeta| \le R$ and $|\arg(\zeta)| \ge \delta$. Then

$$\mathrm{Var}[X] \ge cR^{-2\pi/\delta} n \,.$$



Combined results

Theorem (M.-Sahasrabudhe)

Let $X \in \{0, \dots, n\}$ *with all zeros* ζ *of* $f_X(z)$ *satisfy* $|\arg(\zeta)| \geq \delta$. *Set* $\sigma^2 = \mathrm{Var}(X)$ *and* $X^* = (X - \mathbb{E}X)/\sigma$. *Then*

$$\sup_t |\mathbb{P}(X^* \leq t) - \mathbb{P}(Z \leq t)| \leq \frac{C}{\delta\sigma}\,.$$

If we also assume all roots have $R^{-1} \leq |\zeta| \leq R$ *then*

$$\sigma^2 \geq cnR^{-2\pi/\delta}\,.$$

Combining these two gives

$$\sup_t |\mathbb{P}(X^* \leq t) - \mathbb{P}(Z \leq t)| = O\left(\delta^{-1}R^{-\pi/\delta}n^{-1/2}\right)$$

$$\max_r \mathbb{P}(X = r) = O\left(\delta^{-1}R^{-\pi/\delta}n^{-1/2}\right)$$

Positive coefficients

Direct computation:

$$\mathrm{Var}[X] = \sum \left(\frac{1}{1-\zeta_j} - \frac{1}{(1-\zeta_j)^2} \right) .$$

Not obvious that this is even *non-negative*, let alone large.

X random variable $\implies$ f_X has non-negative coefficients.
$|f_X(z)| \leq f_X(|z|)$ for all z.

Obrechkoff's theorem: "Polynomials with positive coefficients can only have at most $2\alpha n$ roots with in $\arg(\zeta) \in [0, \alpha]|$." Roots are biased away from positive real axis.

The "game"/hard part is understanding how to harness the assumption of positive coefficients.

Proof sketch

Theorem (M.-Sahasrabudhe)

Let $X \in \{0, \ldots, n\}$ be a random variable and let $\zeta_1, \ldots, \zeta_n$ be the roots of f_X. Let

$$\delta = \min_i |\arg(\zeta_i)|,$$

and let

$$X^* := (X - \mu)\sigma^{-1}.$$

Then

$$\sup_{t \in \mathbb{R}} |\mathbb{P}(X^* \leq t) - \mathbb{P}(Z \leq t)| = O\left(\frac{1}{\delta\sigma}\right),$$

where $Z \sim N(0,1)$.

$X \in \{0, \ldots, n\}$, f_X is a polynomial with non-negative coefficients and no zeros in $\Omega := \{z : |\arg(z)| < \delta\}$.

$$\log |f_X(z)|$$

Key fact: If f_X has no zeros in Ω if and only if $\log |f_X(z)|$ is harmonic on Ω.

Fix $\gamma = \delta/2$ and define

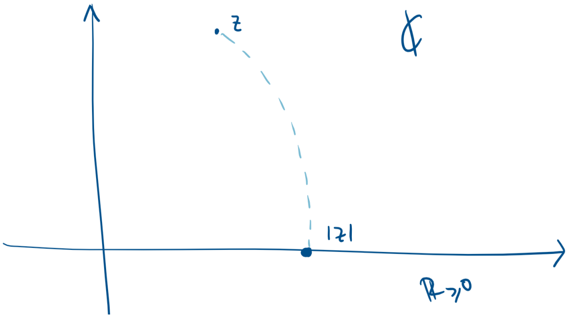
$$\varphi_\gamma(z) := \log |f_X(z)| - \log |f_X(e^{i\gamma} z)|$$

$$\varphi_\gamma(e^w) = \sum_{k \geq 2} a_k \mathrm{Re}(w^k - (w + \gamma i)^k).$$

Key Fact: a_k are (re-normalized) cumulants of X. $a_2 = -\sigma^2/2$
Key Fact: Random variable X is normal if a_2 "dominates" the sequence $(a_k)_k$.

Non-negativity of the coefficients f implies

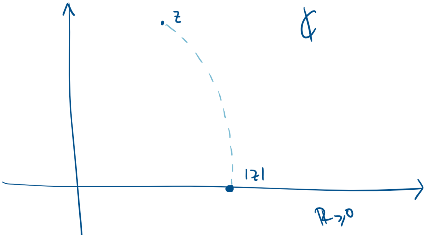
$$\log|f(z)| \leq \log|f(|z|)|$$



Key Fact: $\varphi_\gamma(z) = \log|f(z)| - \log|f(e^{i\gamma}z)| \geq 0$ for $z \in \mathbb{R}^{>0}$

Step 1

Key Fact: $\varphi_\gamma(z) = \log|f(z)| - \log|f(e^{i\gamma}z)| \geq 0$ for $z \in \mathbb{R}^{>0}$.



Lemma (Step 1)

Our function $\varphi_\gamma(z)$ is positive for $|\arg(z)| \leq \delta/4$.

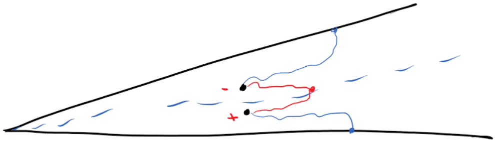
Step 1, continued

f has positive coefficients $\implies \varphi_\gamma(z) = \log|f(z)| - \log|f(e^{i\gamma}z)| \geq 0$ for $z \in \mathbb{R}^{>0}$.

Lemma (Step 1)

Our function $\varphi_\gamma(z)$ is positive for $|\arg(z)| \leq \delta/4$.

Idea: write $z = re^{i\theta}$. Do Poisson integration to write $\log|f(z)|$ as integral along boundary of the sector $\arg(w) \in [0, \theta + \gamma/2]$. Do the same for $\log|f(e^{i\gamma}z)|$ along boundary of $\arg(w) \in [\theta + \gamma/2, 2\theta]$. By symmetry, the integrals of each along the ray $\arg(w) = \theta + \gamma/2$ exactly cancel.



Step 2

$$\varphi_\gamma(e^w) = \sum_{k \geq 2} a_k \mathrm{Re}(w^k - (w + \gamma i)^k).$$

Lemma (Step 2)

There exists $\delta_0 \approx \delta$ so that for all $L \geq 2$,

$$\frac{\sum_{j \geq L} |a_j| \delta_0^j}{\sum_{j \geq 2} |a_j| \delta_0^j} \leq C \cdot 2^{-L}.$$

Key Fact: Random variable X is normal if a_2 “dominates” the sequence $(a_k)_k$.

Step 3

Theorem (Marcinkiewicz)

Let

$$\psi(\xi) := \mathbb{E}_X e^{iX\xi}.$$

If

$$\psi(\xi) = e^{P(\xi)},$$

where $P(\xi)$ is a polynomial then $\deg(P(\xi)) \leq 2$.

$$\varphi_\gamma(e^w) = \sum_{k \geq 2} a_k \mathrm{Re}(w^k - (w + \gamma i)^k).$$

Lemma (Step 3)

If

$$\frac{\sum_{k \geq L} |a_k| \delta_0^k}{\sum_{k \geq 2} |a_k| \delta_0^k} < 1/2$$

for some $\delta_0 > 0, L \geq 1$ *then there exists a real number* $\delta_1 \approx 2^{-L}\delta_0$ *for which*

$$\sigma^2/2 = |a_2| \geq \delta_1^{k-2}|a_k|,$$

for all $k \geq 2$*.*

Work with a different "difference" function: $\log|f(|z|)| - \log|f(z)|$.

Recap of steps

Lemma (Step 1)

Our function $\varphi_\gamma(z) = \log|f(z)| - \log|f(e^{i\gamma}z)|$ *is positive in the sector* $|\arg(z)| \leq \delta/4$.

Define the cumulant sequence (a_j) defined by $\log|f(e^w)| = \sum_j a_j \mathrm{Re}(w^j)$.

Lemma (Step 2)

There exists $\delta_0 \approx \delta$ *so that for all* $L \geq 2$,

$$\frac{\sum_{j \geq L} |a_j| \delta_0^j}{\sum_{j \geq 2} |a_j| \delta_0^j} \leq C \cdot 2^{-L}.$$

Lemma (Step 3)

For sufficiently large L, *there exists a real number* $\delta_1 \approx 2^{-L}\delta_0$ *for which* $|a_2| \geq \delta_1^{k-2}|a_k|$, *for all* $k \geq 2$.

More general context

Properties used were: zero-freeness, $u(z) \leq u(|z|)$, and a bound on growth (for Poisson integration).

So if μ is a probability measure on $\mathbb{C}$ with logarithmic potential $u(z) = \int \log |z - w| \, d\mu(w)$ satisfying:

- $\mu(\{|\arg(\zeta)| \leq \delta\} = 0$
- $u(z) \leq u(|z|)$
- A growth condition on $u(z)$ as $z \to \infty$ in the sector $\{|\arg(\zeta)| \leq \delta\}$

Then $|a_2| \geq (c\delta)^{k-2} |a_k|$.

Variance lower bound

If $X \in \{0, 1, \ldots, n\}$ with f_X zero-free whenever $|\arg(z)| \leq \delta$, $|z| \leq 1/R$ or $|z| \geq R$, then

$$\mathrm{Var}(X) \geq cR^{-2\pi/\delta}\delta^{-1}n$$

(provided $R \geq 1 + \delta$).

Example

There are examples with

$$\mathrm{Var}(X) = \Theta(R^{-\pi/\delta}\delta^{-1}n)\,.$$

History, and another CLT?

Conjecture (Pemantle, 2017)

For $\delta > 0$, let $X_n \in \{0, \ldots, n\}$ be a sequence of random variables with $\sigma_n \to \infty$. If all the roots ζ of f_{X_n} satisfy $|\zeta - 1| > \delta$, then

$$(X_n - \mu_n)\sigma_n^{-1} \to N(0,1),$$

in distribution, as $n \to \infty$.

- (1979) Iagolnitzer and Souillard: Pemantle's theorem is true if $\sigma_n \gg n^{1/3}$ "in the context of the Ising model".
- (2013) Hwang and Zacharovas: Pemantle's theorem is true if all the roots are on the unit circle.
- (2016) Lebowitz, Pittel, Ruelle and Speer: Pemantle's theorem is true if $\sigma_n \gg n^{1/3}$.
- (2018) M.-Sahasrabudhe: Pemantle's conjecture is true $\sigma_n > n^{\varepsilon}$, for any $\varepsilon > 0$.
- (2018) M.-Sahasrabudhe: Pemantle's conjecture is false!

Another CLT

Zero-freeness in sector $\implies$ CLT. Still have a CLT under much more less restrictive conditions.

Theorem (M.-Sahasrabudhe)

Let $X \in \{0, \ldots, n\}$ with $f_X(z)$ zero free in the ball of radius γ around 1. Set $\sigma^2 = \mathrm{Var}(X)$ and $X^* = (X - \mathbb{E}X)/\sigma$. Then

$$\sup_t |\mathbb{P}(X^* \le t) - \mathbb{P}(Z \le t)| \le \frac{C \log n}{\gamma\sigma} .$$

CLT if $\sigma \gg \log n/\gamma$. This is sharp up to C.

Multivariate zero-freeness

$X \in \{0, \ldots, n\}^d$

Probability generating function of X:

$$f_X(z_1, \ldots, z_d) = \sum_{i_1, \ldots, i_d} \mathbb{P}(X = (i_1, \ldots, i_d)) z_1^{i_1} \cdots z_d^{i_d}.$$

f_X is *real-stable* if it has no roots in

$$\mathbb{H} := \{(z_1, \ldots, z_d) \in \mathbb{C}^d : \mathrm{Im}(z_i) > 0, \text{ for all } i\}.$$

Another perspective

Question (Pemantle, '00)

What is the "correct" notion of negatively dependent random variables?

Theorem (Borcea, Brändén, Liggett, '07)

The correct definition is

"$X_1, \dots, X_d$ are negatively dependent random variables if the (multi-variate) probability generating function of $X = (X_1, \dots, X_d)$ is real stable."

Recall real stable means $f_X(z_1, \dots, z_d)$ does not vanish if $\mathrm{Im}(z_j) > 0$ for all j.

Question

What is the limit shape of these distributions?

Conjecture (Ghosh, Liggett, Pemantle, 2017)

For $d \in \mathbb{N}$, let $X_n \in \{0, \ldots, n\}^d$ be a sequence of random variables. If f_{X_n} is real stable and $\mathrm{Cov}(X_n)\sigma_n^{-2} \to A$ with $\sigma_n \to \infty$ then

$$(X_n - \mu_n)\sigma_n^{-1} \to N(0, A)\,.$$

Theorem (M.-Sahasrabudhe)

The Ghosh-Liggett-Pemantle conjecture is true.

Thanks!

Theorem (M.-Sahasrabudhe)

Let $X \in \{0, \dots, n\}$ with $\mathbb{P}(X = 0)\mathbb{P}(X = n) > 0$ and suppose all zeros ζ of $f_X(z)$ satisfy $R^{-1} \leq |\zeta| \leq R$ and $|\arg(\zeta)| \geq \delta$. Then

$$\mathrm{Var}[X] \geq cR^{-2\pi/\delta} n.$$

$$|\mathbb{P}(X^* \leq t) - \mathbb{P}(Z \leq t)| = O\left(\delta^{-1} R^{-\pi/\delta} n^{-1/2}\right)$$

$$\max_r \mathbb{P}(X = r) = O\left(\delta^{-1} R^{-\pi/\delta} n^{-1/2}\right)$$

(with $X^* = (X - \mu)\sigma^{-1}$ and $Z \sim N(0, 1)$).