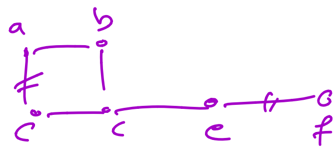
March 12 Seminar on Geometry, Probability and Computing.

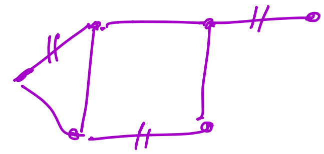
The Matching Polytope has Exponential Extension Complexity. by Thomas Rothvoss

- Matching Polytope $G = (V, E)$

$M \subseteq E$ be a matching ("ind. edge set") they do not share a common vertex.



$M \subseteq E$ is a perfect matching if the matching and 'matches' all vertices of the graph



· $\chi_M \in \mathbb{R}^{|E|} \in \{0,1\}^{|E|}$

$$\langle \chi_M, e_e \rangle = \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{otherwise.} \end{cases}$$

· The matching polytope $P_M = \text{conv}\{\chi_M : M \text{ is a matching of } G\}$

- $x_e \geq 0$, $\forall e \in E$ (1)
- For every edge one can have at most one adjacent edge in the matching

$U \subseteq V$, $\boxed{\delta(U) = \text{"the set of edges that have exactly one point in } U\text{"}}$.

$\delta: V \to E$

$$\sum_{e \in \delta(v)} x_e \leq 1 \qquad \forall v \in V. \quad (2)$$

- If I consider a subgraph defined by a matching, then $U \subseteq V$ with odd cardinality must have no more than $\frac{|U|-1}{2}$ edges

$$(3) \quad \sum_{e \in \mathbb{E}(U)} x_e \leq \frac{|U|-1}{2}, \quad |U| \text{ odd}, \ U \subseteq V$$

the edges defined on the subgraph defined by U

Theorem (Edmond's)

$$P_M = \{ x \in \mathbb{R}^{|E|}, \ x \text{ satisfies } (1), (2), (3) \}.$$

P_{PM} it is a facets of P_M.

Complexity

P polytope

$P = \operatorname{conv}\{z_1, \dots, z_r\}$

$P = \{x : Ax \le b\}$ $\quad A = \begin{pmatrix} A_1 \\ \vdots \\ A_f \end{pmatrix}$

f the number of facets

· **Example**

· $B_1^n = \{x \in \mathbb{R}^n : \sum_{i=1}^n |x_i| \le 1\}$

2^n facets

· $Q = \{(x, y) \in \mathbb{R}^{2n} : \sum y_i = 1,$
$-y_i \le x_i \le y_i, \; i = 1, \dots, n\}$

$\mathbb{P}_{\{x\}} Q = B_1^n$.

· $\Pi_n = \operatorname{conv}\{(\pi(1), \dots, \pi(n)), \; \pi \in \Pi.\}$

It has $2^n - 2$ facets

$\exists$ Q in $\mathbb{R}^d$ with at most $\sim d$ facets
$d \sim n \log n$

and T linear $\quad \Pi_n = TQ$. (Goemans)

· Q polytope $\quad |Q|$ the number of facets

Question: if P fixed polytope $\exists ?$ Q lift.

and F affine and a $\exists$ linear map

such that $P = T(Q \cap F)$

with $|Q|$ as small as possible

$xc(P)$ $\quad n = |V|$

<u>Theorem</u> The extension complexity of the perfect matching polytope is of a complete graph, is 2^{cn}

<u>Cor</u> For odd n the conv. P_{TSP} is again 2^{cn}.

- Polytope $P = \text{conv}\{z_1, \dots, z_v\}$ $\quad f$ facets
  $= \{x \in \mathbb{R}^n : Ax \le b\}$ $\quad A = \begin{pmatrix} A_1 \\ \vdots \\ A_f \end{pmatrix}$

- <u>Slack matrix</u> of P
  $S = S_P \in \mathbb{R}_{\geq 0}^{f \times v}$ such that $S_{ij} = b_i - A_i z_j$

- <u>Non-negative rank</u> of a matrix S:

$$rk_+(S) = \min\{r \mid \exists\, U \in \mathbb{R}_{\geq 0}^{f \times r}, V \in \mathbb{R}_{\geq 0}^{r \times n} : S = UV\}.$$

Theorem (Yiannakakis) ('91)

$$xc(P) = rk_+(S_P)$$

Proof Let $P = \text{conv}\{z_1, \ldots, z_v\}$
$= \{x \in \mathbb{R}^v : Ax \leq b\}$

$$S = (b - Az_1, \ldots, b - Az_k) = L\, R \quad ①$$

Define $Q = \{(x, y) \in \mathbb{R}^{v+r},\ y_i \geq 0$
$Ax + Ly = b\}$

π. orthogonal projection on "x".

$R = (R_1, \ldots, R_v)$.

Then. $b - Az_j = L\, R_j \rightsquigarrow Az_j + LR_j = b$

So $z_j \in \pi \cdot (Q) \Longrightarrow P \subseteq \pi(Q)$

Let $x' \notin P \Longrightarrow \exists\, i : A_i x' > b_i$

So $\underbrace{A_i x'}_{} + \underbrace{L_i y}_{\geq 0} > b_i \quad \forall y$

$\Longrightarrow \nexists\, y \in \mathbb{R}^r_+ : (x', y) \in Q.$

$\Longrightarrow P \supseteq \pi(Q)$

$(\Longleftarrow)$ <u>Lemma</u> If $P = \{x \in \mathbb{R}^n : Ax \le b\}$
and $\langle x, c \rangle \le \delta \quad \forall \; x \in P$

$\Longrightarrow \; \exists \; y \in \mathbb{R}^k$: $\quad y \ge 0$

$y^T A = c^T$

$\langle y, b \rangle = \delta$

Let $\underline{Q = \{(x, y) : Bx + Cy \le d\}} \subseteq \mathbb{R}^{n+e}$

$$B = \begin{pmatrix} B_1 \\ \vdots \\ B_r \end{pmatrix}, \qquad C = \begin{pmatrix} c_1 \\ \vdots \\ c_r \end{pmatrix}$$

π orthogonal projection $\quad P = \pi(Q)$

$$(A_i, 0) \begin{pmatrix} x \\ y \end{pmatrix} \le b_i \qquad i \le m.$$

Lemma $\Longrightarrow \; \exists \; L_1, \dots, L_m \in \mathbb{R}^r$: $\quad L_i \ge 0$

$\underline{L_i (B, C)} = \underline{(A_i; 0)}$

$\underline{L_i d = b_i}$

$\forall \; z_j \in P \quad \exists \; y_j : \quad (z_j, y_j) \in Q.$

<u>Define</u> $\quad R_j = (d - Bz_j - Cy_j). \quad j = 1, \dots, k$
non-negative

$$L = \begin{pmatrix} L_1 \\ \vdots \\ L_m \end{pmatrix}, \qquad R = (R_1, \dots, R_k)$$

$$L_i R_{y_j} = L_i (d - Bz_j - Cy_j)$$
$$= L_i d - L_i B z_j - 0 =$$
$$= b_i - A_i z_j = S_{ij}$$

<u>Lemma</u> (Hyperplane separation lower bound) (Fiorini).

Let $S = S_P$ slack matrix of P.

Then $xc(P) \geq \dfrac{\langle W, S_P \rangle}{\|S\|_\infty \cdot \alpha(W)}$ for all W

$\alpha(W) = \max\{ \langle W, R \rangle,\ R \text{ rank } 1,\ R \in \{0,1\}^{f \times v} \}$

<u>Proof</u> Let $r = rk_+(S)$. By [Y], $xc(P) = rk_+(S)$

$\exists\ R_1, \dots, R_r : S = \sum_{i=1}^{r} R_i$, R_i non-negative rank 1.

$$\frac{R_i}{\|R_i\|_\infty} \in [0,1]^{f \times v}$$

$$\langle W, S \rangle = \sum_{i=1}^{r} \langle W, \frac{R_i}{\|R_i\|_\infty} \rangle \|R_i\|_\infty$$

$$\leq \underbrace{\max\{\langle W, R\rangle,\ R \in [0,1]^{f\times v}, \text{ rank } 1\}}$$

$$\cdot \sum_{i=1}^{r} \|R_i\|_\infty \leq \alpha(W)\, r\, \|S\|_\infty$$

Remark $\operatorname{conv}\{R \in [0,1]^{f \times v} : \operatorname{rank} R \leq 1\}$

$= \operatorname{conv}\{R \in \{0,1\}^{f \times v} : \operatorname{rank} R \leq 1\}$.

Back to the matching polytope

- $|V| = n = 3m(k-3) + 2k$ m odd.
  $k << n, m$

$\mathcal{U} = \{ U \subseteq V : |U| = t \}$ $t = \frac{m+1}{2}(k-3) + 3$

$\mathcal{M} = \{$all perfect matchings$\}$

$Q_\ell = \{ (U, M) \in \mathcal{U} \times \mathcal{M} : |\delta(U) \cap M| = \ell \}$

μ_ℓ the uniform measure on Q_ℓ

Then Define the matrix $W \in \mathbb{R}^{u \times m}$

$$W_{UM} = \begin{cases} -\infty & |\delta(U) \cap M| = 1 \\ \frac{1}{|Q_3|} & |\delta(U) \cap M| = 3 \\ -\frac{1}{k-1}\frac{1}{|Q_k|} & |\delta(U) \cap M| = k \\ 0 & \text{otherwise} \end{cases}$$

$\boxed{\text{Recall} \quad S_{UM} = |M \cap \delta(U)| - 1}$

$$\langle w, s \rangle = 0 + \underbrace{(3-1)\,|Q_3|\,\frac{1}{|Q_3|}}_{2} - \underbrace{(k-1)\,|Q_k|\,\frac{1}{k-1}\,\frac{1}{|Q_k|}}_{1}$$

$$= 1.$$

<u>**Lemma**</u> $\forall$ k odd ≥ 3 and any $\underline{R = U \times M}$ with $\underline{\mu_1(R) = 0}$, then

$$\mu_3(R) \leq \frac{400}{k^2}\,\mu_k(R) + 2^{-\delta m}, \quad \delta = \delta(k).$$

If assume Lemma.

$$\langle w, R \rangle = \mu_3(R) - \frac{1}{k-1}\,\mu_k(R)$$

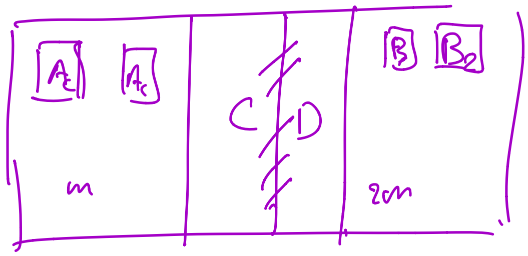
$$\leq \underbrace{\left(\frac{400}{k^2} - \frac{1}{k-1}\right)}_{<0} \mu_k(R) + 2^{-cm}.$$

if $k \geq 501$

So $$xc(P_{\mathrm{cor}}) \geq \frac{\langle w, s \rangle}{\|S\|_\infty \max\{\langle w, R \rangle,\ R \text{ rectangle}\}}$$

$$\geq \frac{1}{n \cdot 2^{-\delta m}} \sim e^{+cm}$$

vertices



$|C| = k$

$|D| = k$

$|A_k| = (k-3)$

$= (k-3)$