The k-set problem: general shapes and random point sets

Luis Rademacher

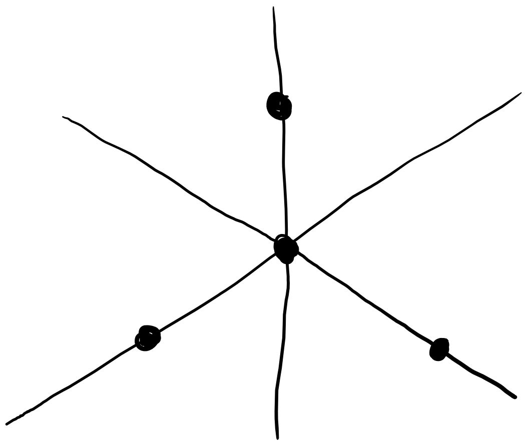
UC Davis

Joint work with Brett Leroux

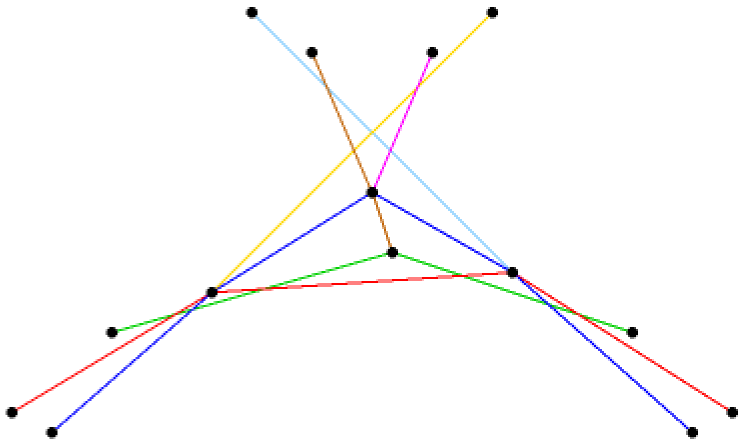
The k-set problem

Halving line problem

- [Simmons, before 1971]: Proposed the problem.
- [Straus] First lower bound.
- [L. Lovász] *On the number of halving lines*, Ann. Univ. Sci. Budapest, Eötvös, Sect. Math. 14:107-108, 1971.
- [Edelsbrunner Welzl] Rediscovered in connection with complexity of search problems in computational geometry.
- P: a finite set of points in R^2.
- **Halving line** of P: a line through two points in P that splits the rest in half.
- **Halving line problem**: Let $e(n)$ =“the maximum number of halving lines over sets of n points”. How does $e(n)$ grow?
- As a geometric graph: **Halving edge graph** $G = (V, E)$, where $V = P$ and place an edge between pairs of vertices determining a halving line.
- Assume from now on that point sets are in general position: no three on a line.



e(4)=3



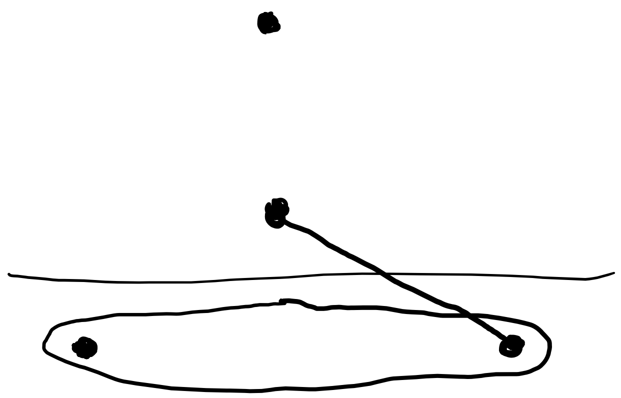
(figure by Jeff Erickson)

A generalization: k-edges and the k-edge graph

- P: a finite set of points in R^2.
- **k-edge of $\boldsymbol{P}$**: a pair of points $u, v \in P$ such that the line through them has k points on one side.
- Let $e(k, n) =$"the maximum number of k-edges over sets of n points". How does $e(k, n)$ grow?
- **k-edge graph**: $G = (V, E)$ where $V = P$ and $E =$ {k-edges}.
- Halving edge = "k-edge with $k = (n - 2)/2$."

A variation: the k-set problem

- P: a finite set of point on the plane.
- **k-set of $\boldsymbol{P}$**: a subset of k points that can be separated from the rest by a line.
- **k-set problem**: Let $a(k,n)$ =“the maximum number of k-sets over sets of n points”. How does $a(k,n)$ grow?
- Proposition: $a(k,n) = e(k-1,n)$.
  Proof idea:
  - Take a line that defines a k-set with $k = n/2$.
  - Rotate it clockwise as much as possible without crossing points.
  - This gives a halving edge and is a bijection.



$k = 2, n = 4$

Some bounds for $e(n)$ (max halving lines)

- Asymptotic upper bounds*:
  - $O(n^{3/2})$ [Lovász] Proof soon.
  - $O(n^{4/3})$ [Dey] Proof soon.
- Asymptotic lower bounds:
  - $\Omega(n \log n)$ [Straus] Recursive construction (right).
  - $n\, e^{\Omega(\sqrt{\log n})}$ [Tóth] More complicated recursive construction.
- **Conjecture [Erdős Lovász Simmons Straus]**: truth is close to the lower bound, expect $O(n^{1+\epsilon})$ for all $\epsilon > 0$.

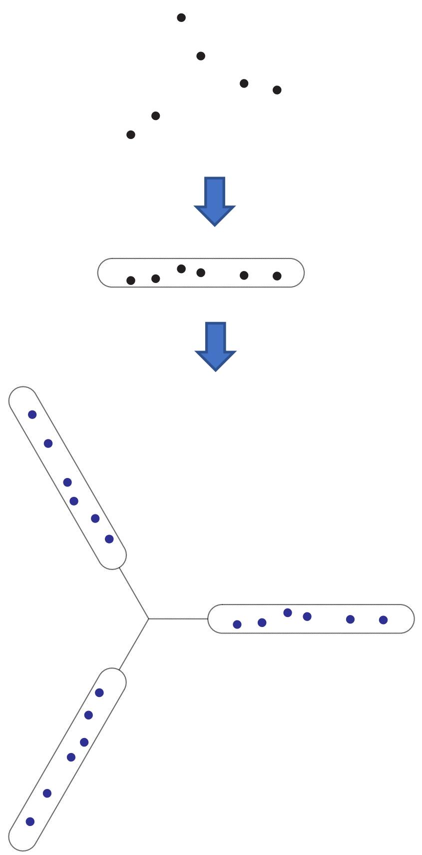
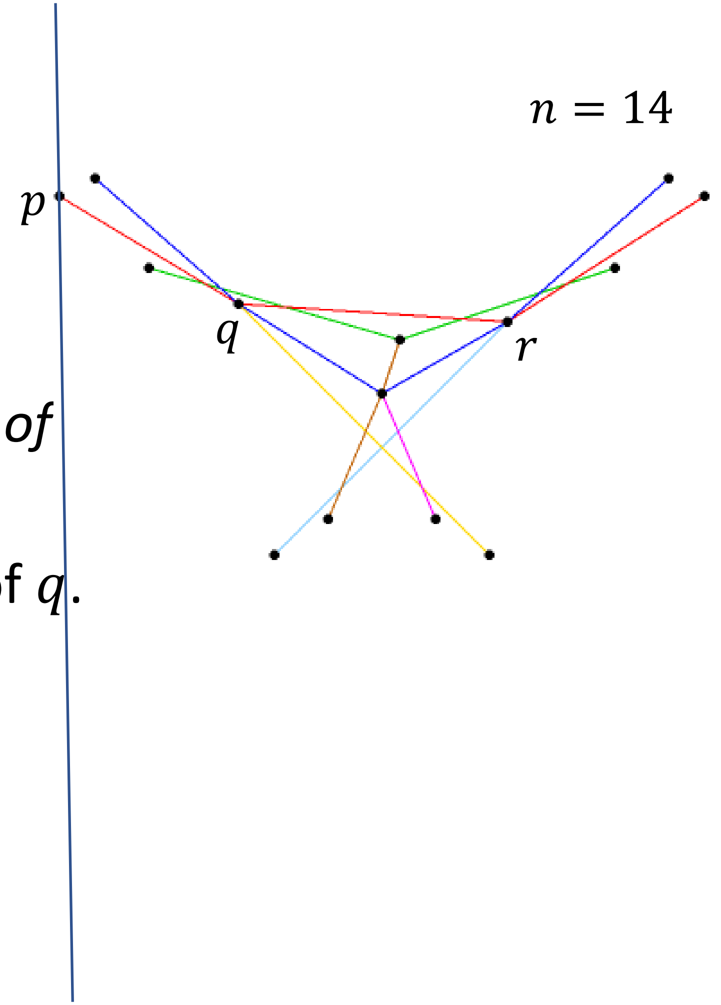


figure from
[Ábrego Fernández-Merchant Salazar]

Structure of k-edge graph: Convex chains

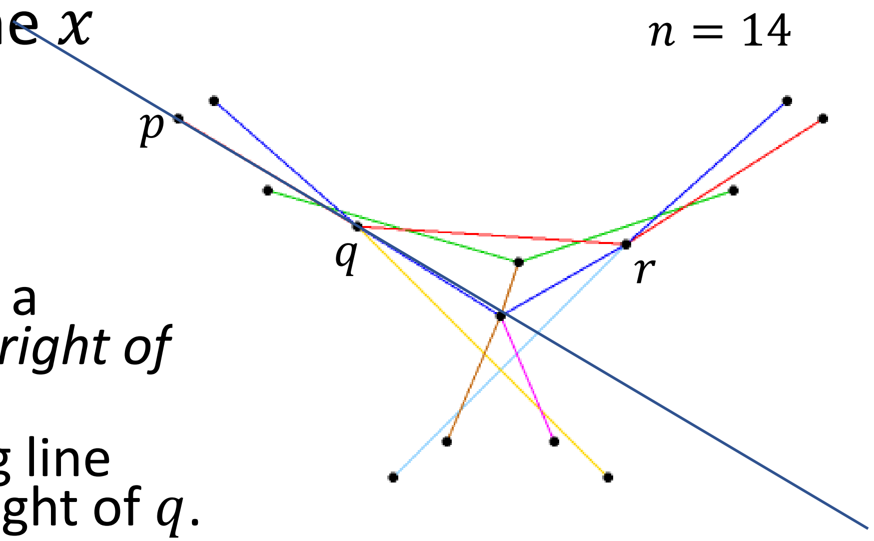
- Assume n is even and no pair of points with same x coordinate, w.l.o.g. (rotate if needed).
- For each $p \in \{n/2 \text{ leftmost points}\}$:
  - Draw a vertical line through p.
  - Rotate it counterclockwise around p until it becomes a halving line. This defines a halving edge (p, q) *to the right of* p*.*
  - Continue rotating around q until it becomes a halving line again. This defines a new halving edge (q, r) to the right of q.
  - Continue rotating the line until it becomes vertical again (180° rotation).
  - The union of the picked halving edges is a **convex chain**.

Thm [Dey]: This **partitions** all halving edges into $n/2$ convex chains.



Structure of k-edge graph: Convex chains

- Assume n is even and no pair of points with same x coordinate, w.l.o.g. (rotate if needed).
- For each $p \in \{n/2 \text{ leftmost points}\}$:
  - Draw a vertical line through p.
  - Rotate it counterclockwise around p until it becomes a halving line. This defines a halving edge (p, q) *to the right of* p.
  - Continue rotating around q until it becomes a halving line again. This defines a new halving edge (q, r) to the right of q.
  - Continue rotating the line until it becomes vertical again (180° rotation).
  - The union of the picked halving edges is a **convex chain**.

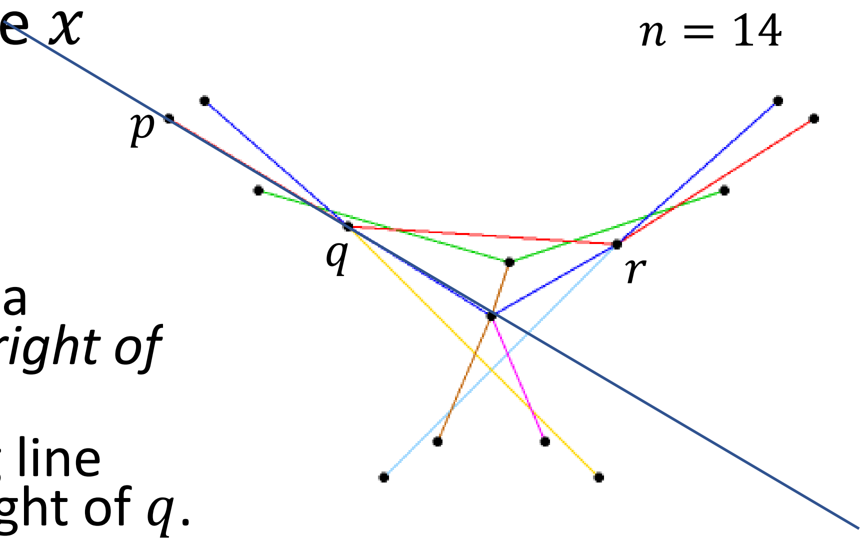


Thm [Dey]: This **partitions** all halving edges into $n/2$ convex chains.

Structure of k-edge graph: Convex chains

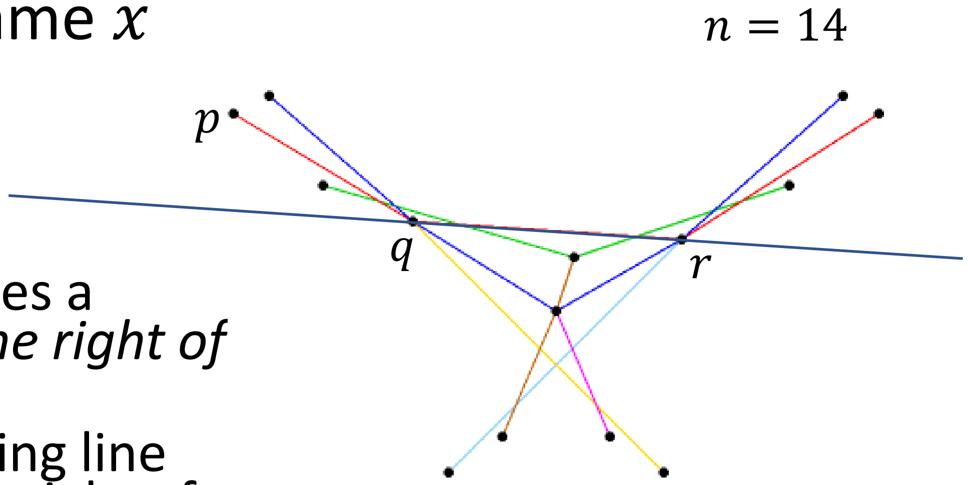
- Assume n is even and no pair of points with same x coordinate, w.l.o.g. (rotate if needed).
- For each $p \in \{n/2 \text{ leftmost points}\}$:
  - Draw a vertical line through p.
  - Rotate it counterclockwise around p until it becomes a halving line. This defines a halving edge (p, q) *to the right of p.*
  - Continue rotating around q until it becomes a halving line again. This defines a new halving edge (q, r) to the right of q.
  - Continue rotating the line until it becomes vertical again (180° rotation).
  - The union of the picked halving edges is a **convex chain**.

Thm [Dey]: This **partitions** all halving edges into $n/2$ convex chains.



Structure of k-edge graph: Convex chains

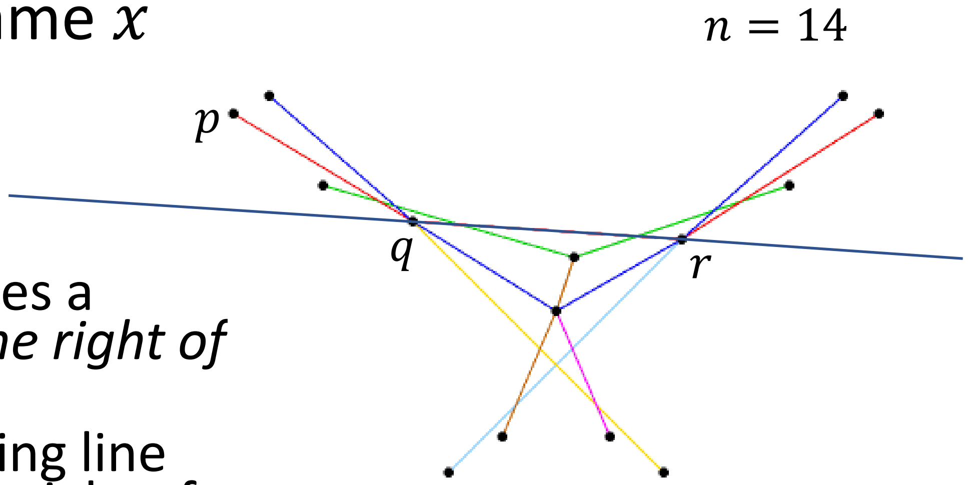
- Assume n is even and no pair of points with same x coordinate, w.l.o.g. (rotate if needed).
- For each $p \in \{n/2 \text{ leftmost points}\}$:
  - Draw a vertical line through p.
  - Rotate it counterclockwise around p until it becomes a halving line. This defines a halving edge (p, q) *to the right of* p.
  - Continue rotating around q until it becomes a halving line again. This defines a new halving edge (q, r) to the right of q.
  - Continue rotating the line until it becomes vertical again (180° rotation).
  - The union of the picked halving edges is a **convex chain**.



Thm [Dey]: This **partitions** all halving edges into $n/2$ convex chains.

Structure of k-edge graph: Convex chains

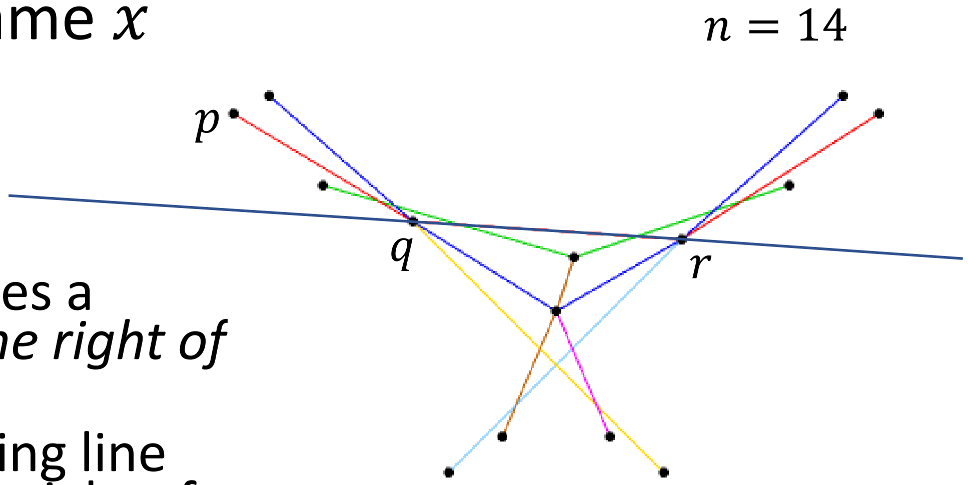
- Assume n is even and no pair of points with same x coordinate, w.l.o.g. (rotate if needed).
- For each $p \in \{n/2 \text{ leftmost points}\}$:
  - Draw a vertical line through p.
  - Rotate it counterclockwise around p until it becomes a halving line. This defines a halving edge (p, q) *to the right of* p.
  - Continue rotating around q until it becomes a halving line again. This defines a new halving edge (q, r) to the right of q.
  - Continue rotating the line until it becomes vertical again (180° rotation).
  - The union of the picked halving edges is a **convex chain**.



Thm [Dey]: This **partitions** all halving edges into $n/2$ convex chains.

Structure of k-edge graph: Convex chains

- Assume n is even and no pair of points with same x coordinate, w.l.o.g. (rotate if needed).
- For each $p \in \{n/2 \text{ leftmost points}\}$:
  - Draw a vertical line through p.
  - Rotate it counterclockwise around p until it becomes a halving line. This defines a halving edge (p, q) *to the right of* p.
  - Continue rotating around q until it becomes a halving line again. This defines a new halving edge (q, r) to the right of q.
  - Continue rotating the line until it becomes vertical again (180° rotation).
  - The union of the picked halving edges is a **convex chain**.

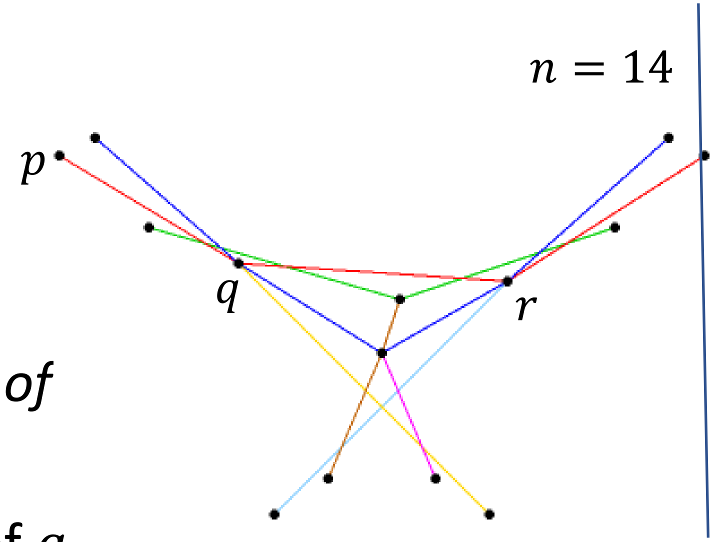


Thm [Dey]: This **partitions** all halving edges into $n/2$ convex chains.

Structure of k-edge graph: Convex chains

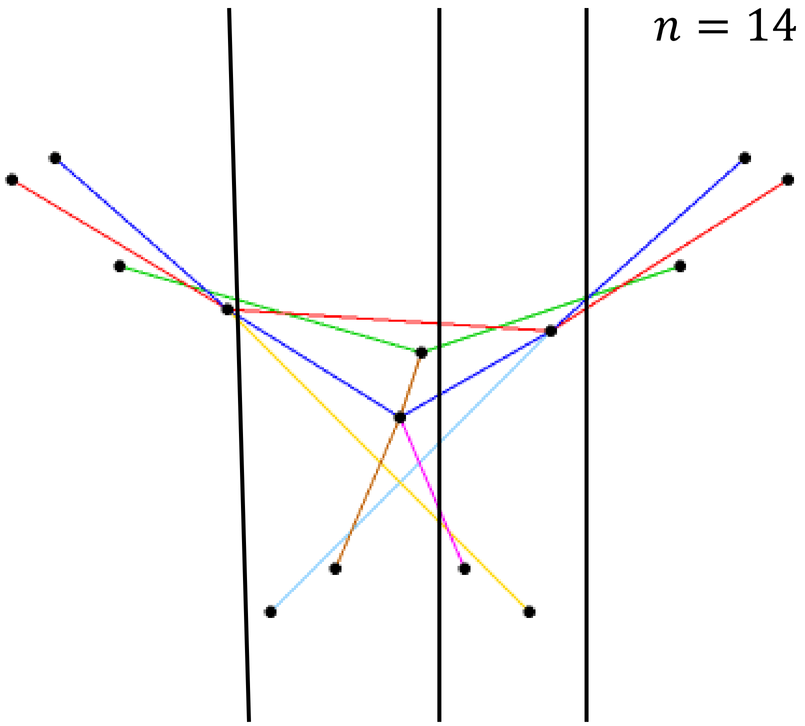
- Assume n is even and no pair of points with same x coordinate, w.l.o.g. (rotate if needed).
- For each $p \in \{n/2 \text{ leftmost points}\}$:
  - Draw a vertical line through p.
  - Rotate it counterclockwise around p until it becomes a halving line. This defines a halving edge (p, q) *to the right of p.*
  - Continue rotating around q until it becomes a halving line again. This defines a new halving edge (q, r) to the right of q.
  - Continue rotating the line until it becomes vertical again (180° rotation).
  - The union of the picked halving edges is a **convex chain**.

Thm [Dey]: This **partitions** all halving edges into $n/2$ convex chains.



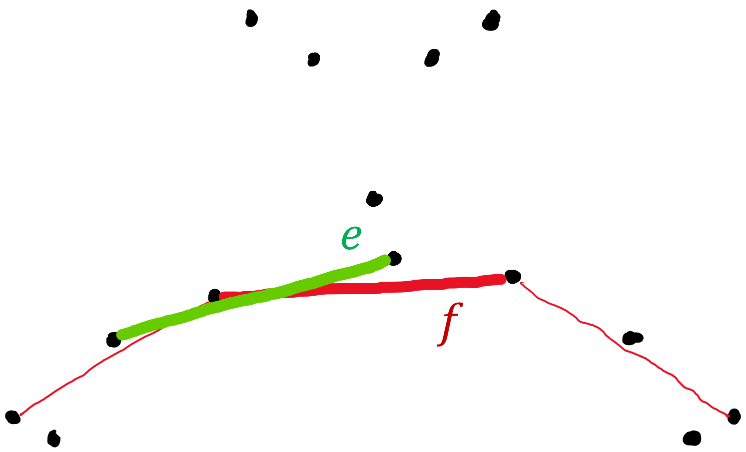
Vertical lines argument: $e(n) \leq O(n^{3/2})$ [Lovász]

- Assume no two points have the same x coordinate w.l.o.g.
- Left to right, draw a vertical line every $\sqrt{n}$ points. This is $\approx \sqrt{n}$ lines partitioning plane.
- Count halving edges as follows:
  - Edges within parts:
    - Each part contains $\sqrt{n}$ points $\Rightarrow \leq n$ edges per part, $\sqrt{n}$ parts implies $\leq n^{3/2}$ edges.
  - Edges across parts:
    - Each line crosses each convex chain at most once: $\leq n\sqrt{n}$ edges.
  - Total $\leq 2n^{3/2}$.



The crossing lemma $\Rightarrow e(n) \leq O(n^{4/3})$ [Dey]

- Explicit "topological" aspect: crossing lemma.
- **Crossing**: an intersecting pair of edges in a geometric graph (intersection not at endpoints).
- **Lemma (Crossing lemma)**: Draw a graph $G = (V, E)$ on the plane. If $|E| > 4|V|$, then the number of crossings is at least $|E|^3/64|V|^2$.
- **Thm [Dey]**: $e(n) \leq O(n^{4/3})$.
  Proof idea:
  - **Claim:** The halving edge graph has $\leq n^2/2$ crossings (using additional ideas by [Har-Peled]). Proof:
    - Consider separately convex chain decomposition and concave chain decomposition.
    - Count every crossing of edges (e, f) as e in a convex chain and f in a concave chain.
      But a convex chain and a concave chain can cross at most twice.
      $\Rightarrow$ number of crossings is at most $2\,(n/2)(n/2) = n^2/2$.
  - Crossing lemma $\Rightarrow$ crossings $\geq |E|^3/64n^2 \Rightarrow |E| \leq O(n^{4/3})$.



Proof of crossing lemma: random sampling

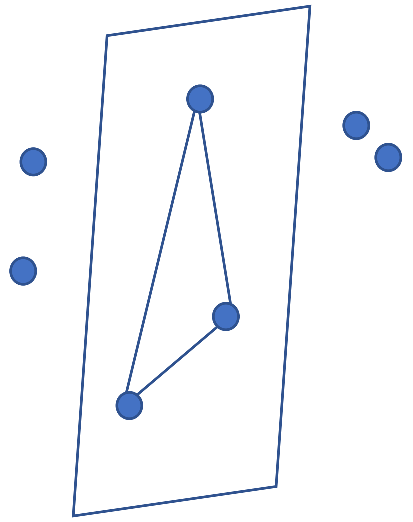
Lemma (Crossing lemma) [Ajtai Chvátal Newborn Szemerédi] [Leighton]: Draw a graph $G = (V, E)$ on the plane. If $|E| > 4|V|$, then the number of crossings is at least $|E|^3/64|V|^2$.
Proof idea:

- Draw G on the plane with v vertices, e edges and c crossings.
- Basic bound: $c \geq e - 3v$.
  - Euler's formula ⇒ a planar graph with v' vertices and e' edges satisfies $e' \leq 3v'$.
    Starting from G, repeatedly "remove one edge from a crossing" until there are no crossings, resulting in $v' = v$ vertices and e' edges.
    We have $e' \geq e - c \Rightarrow e - c \leq e' \leq 3v' \Rightarrow c \geq e - 3v$.
- "Amplify" bound by applying it to a **random subgraph** of G:
  - Pick each vertex with probability p. E(#vertices) $= pv$, E(#edges) $= p^2 e$, E(#crossings) $= p^4 c$.
  - Basic bound gives $p^4 c \geq p^2 e - 3pv$.
  - Take $p = 4v/e$ (say, optimize over p) to get $c \geq e^3/64v^2$.

The d-dimensional case

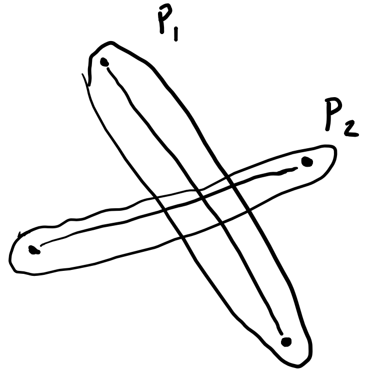
- P: a finite set of points in R^d in general linear position (any $d+1$ or fewer are affinely independent).
- **halving facet**: d points that determine a hyperplane (and simplex) that splits the rest in half.
- **k-facet**: d points that determine a hyperplane (and simplex) that has k points on one side.
- Some known bounds on max # of halving facets:
  - $O(n^d)$ (clearly)
  - $O(n^{d-\epsilon_d})$ where $\epsilon_d \approx 1/d^d$ [Alon Bárány Füredi Kleitman] (via colorful Tverberg theorem)
  - $n^{d-1}\, e^{\Omega(\sqrt{\log n})}$ [Seidel]

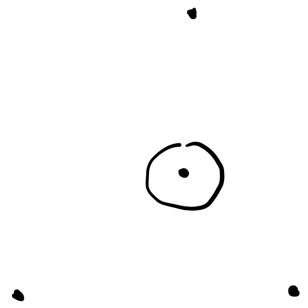
$d = 3, n = 7, k = 2$



"Topological" aspect: colorful Tverberg theorem

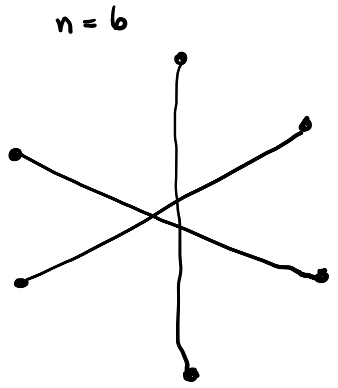
- **Thm [Radon]**: Any set of at least $d + 2$ points in R^d has a partition into parts P_1, P_2 so that $\mathrm{conv}(P_1) \cap \mathrm{conv}(P_2) \neq \emptyset$.
- **Thm [Tverberg]**: Radon but partition into r parts, different bound.
- **Thm (Colorful Tverberg) [Bárány Füredi Lovász] [Živaljević Vrećica]**: Tverberg but points are colored with t colors and the r subsets are disjoint (not a partition) and each subset picks one point from each color. Different bound.

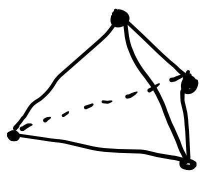




Problem easier for special point sets?

- Points in convex position in R^2? Yes, **always** $n/2$ (“diagonals of a polygon”, exact count!).
- Points in convex position in R^3? Yes, **always** $(n-1)^2/4$ (see below).
- Points in convex position in R^d? Unknown. But some improvement beyond general case possible, later.
- “Correct” generalization of convex position from R^2, R^3 to R^d: **neighborly** point sets.
  - **Def:** A finite set of points P in R^d is **neighborly** if every subset of $\lfloor d/2 \rfloor$ or less points of P determines a face of $\mathrm{conv}(P)$ (has a supporting hyperplane containing exactly those points).
    - If P has $> d+1$ points, then “$\lfloor d/2 \rfloor$” is largest possible by Radon’s theorem.
  - By **Upper Bound Theorem/Dehn-Sommervile equations**: P is neighborly (and in general position) ⇒ # of facets of $\mathrm{conv}(P)$ is **determined** by n and d. Explicit formula.
  - Note that a facet is a k-facet for $k = 0$.
  - Random sampling type technique from [Clarkson Shor]: P is neighborly (and in general position) ⇒ # of k-facets of P is **determined** by n, k and d. Explicit formula [Andrzejak Welzl] [Wagner].

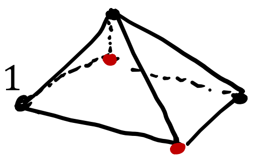




Every 1,2 and 3 points form a face

Not every 2 points form a face

$d = 3$,
$\lfloor d/2 \rfloor = 1$



Application of random sampling technique [Clarkson Shor]

Prop: P is neighborly (and in general position) ⇒ # of k-facets of P is **determined** by n, k and d. Explicit formula [Andrzejak Welzl] [Wagner].

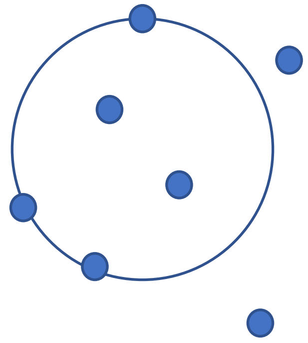
Proof idea:

- Subsets of neighborly point sets are neighborly.
- By previous theorem, # of facets of every subset of P is determined by its # of points and d.
- Let Q be P with a random point removed.
  Let $R \subseteq P$ be a fixed 1-facet of P. $P(R \text{ is a facet of } Q) = 1/n$.
  Let $S \subseteq P$ be a fixed facet of P. $P(S \text{ is a facet of } Q) = (n-d)/n$.
- $E(\text{\# of facets of } Q) = \frac{1}{n}\,(\text{\# of } 1\text{−facets of } P) + \frac{n\text{−}d}{n}(\text{\# of facets of } P)$.
- This determines # of 1-facets of P.
- Similar argument for # of 2-facets of P, # of 3-facets of P, etc.

The k-set problem: general shapes and random point sets

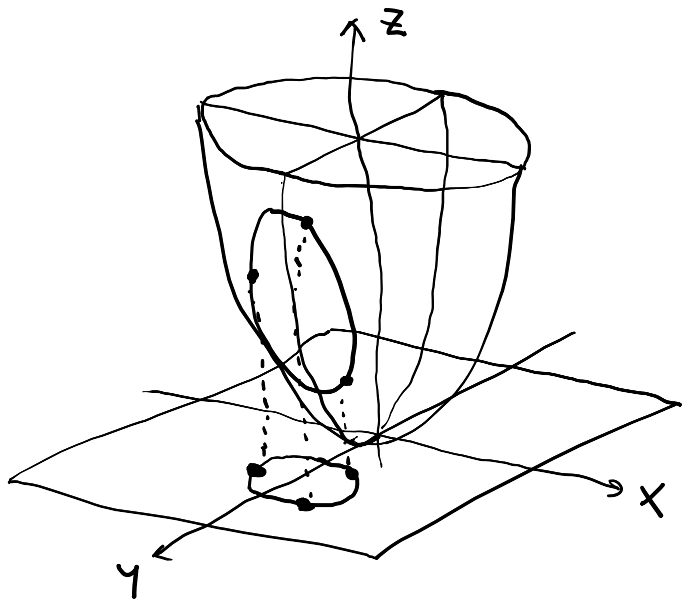
The k-set problem for general set systems

- **Halving line problem**: asymptotics of max # of pairs of points determining a line that splits the rest in half.
- General set systems version: replace lines by another family of shapes determined by a fixed small number of points.
- Example:
  Thm [Lee] [Ardila]: Let $P \subseteq R^2$ be a finite set of $2n+1$ points in general position (no 4 on a circle, no 3 on a line). Then the # of ways in which a circle going through 3 points splits the rest in half is n^2.
- Circles problem is easier than lines: **exact count**. (like neighborly point set case). Any set of n points in general position has the same count.



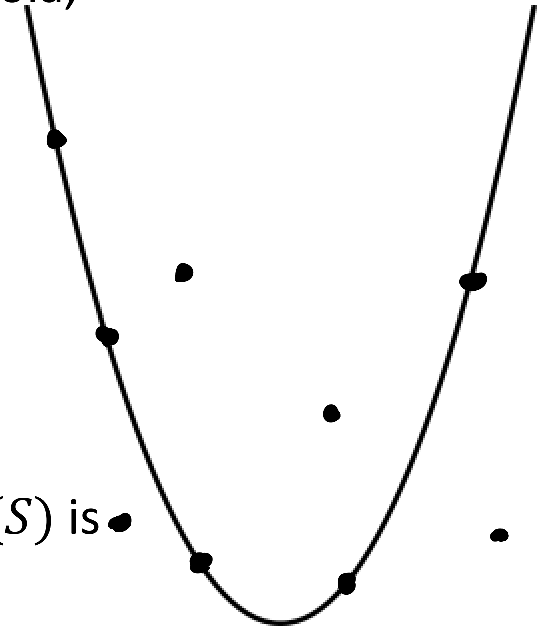
Exact count for circles

- **Thm [Lee] [Ardila]**: Let $P \subseteq R^2$ be a finite set of $2n+1$ points in general position (no 4 on a circle, no 3 on a line). Then the # of ways in which a circle going through 3 points splits the rest in half is n^2.
- Proof idea:
  - Map points to R^3: $(x, y) \mapsto (x, y, x^2+y^2)$.
  - Mapped set of points is in convex=neighborly position.
    - Convex position because map embeds R^2 as a paraboloid, a strictly convex surface.
    - Neighborly because convex=neighborly position in R^3.
  - Halving facet in $R^3 \leftrightarrow$ halving circle in R^2. Namely, $Ax+By+Cz=D \leftrightarrow Ax+By+C(x^2+y^2)=D$
  - Use exact count of halving facets for neighborly.



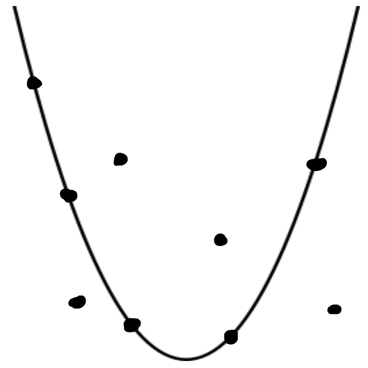
Our results: Exact count for conic sections

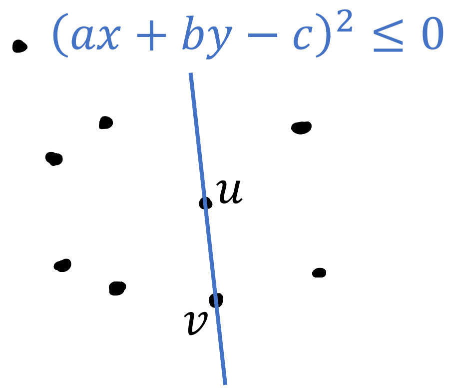
- Want: replace halving lines by "halving conic sections." (conic section=set in R^2 satisfying $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, generically a parabola, hyperbola or ellipse)
- How? Idea: set systems/shapes/surfaces induced by maps.
- Map $\varphi: R^2 \to R^5$, $\varphi(x, y) = (x^2, xy, y^2, x, y)$.
  Then a hyperplane $\{v \in R^5 : \ a \cdot v = b\}$ induces 3 regions in R^2:
  - $a \cdot \varphi(x, y) < b$,
  - $a \cdot \varphi(x, y) > b$ and
  - a "boundary/surface" $a \cdot \varphi(x, y) = b$.
- "Boundary" is a conic section.
- **Def (Halving conic section of $\boldsymbol{P \subseteq R^2}$)**: set of 5 points S of P such that $\varphi(S)$ is a halving facet (determines a halving hyperplane) of $\varphi(P)$ in R^5.
- **Thm [our work]**: Any set of n points in R^2 in general position has $(n-1)^2(n-3)^2/64$ halving conic sections.



Our results: Exact count for conic sections

- **Thm [our work]**: Any set of n points in R^2 in general position has $(n-1)^2(n-3)^2/64$ halving conic sections.
  Proof idea:
  - Map points to R^5: $(x, y) \mapsto (x^2, xy, y^2, x, y)$.
  - **Proposition (mapped set of points is in neighborly position):** Assume a finite set of points $S \subseteq R^2$ is in general linear position. Then $\varphi(S)$ is neighborly.
    Proof: Let $u, v \in S$. We need to find a conic section inequality "passing" though those two points and with all other points on one side. Use line $ax + by = c$ through u and v. The desired inequality is $(ax + by - c)^2 \leq 0$.
  - Halving facet in R^5 $\leftrightarrow$ halving conic section in R^2.
  - Use exact count of halving facets for neighborly point sets.





Our results: Exact count for some polynomial families (even degree homogeneous)

- **Thm (neighborly) [our work]:** If m is even and $S \subseteq R^2$ is in general position (no two points on a common line through the origin), then the image of S through the map of all monomials of degree m is neighborly.
  Proof idea: Like for conic sections, find support hyperplanes by constructing explicit polynomials.
- **Thm (exact count) [our work]:** Assume m is even. Any set of $2n + m + 1$ points of R^2 in general position with respect to degree m homogeneous polynomials has exactly $2\binom{k+m/2}{m/2}\binom{n-k-m/2-1}{m/2}$ "degree m homogeneous polynomials"-k-facets.
  Proof idea: Like conic sections.
- Why does m need to be even? # of monomials in map φ is $m + 1$.
  If m odd, # of monomials in map φ is even (and vice-versa) $\Rightarrow$ embedding dimension is even
  $\Rightarrow$ neighborliness is "stronger" requirement ($\lfloor d/2 \rfloor$).
  E.g. "all pairs of points are a face" in R^4 and same "with more room" in R^5, not stronger. Does not work.

Our results: Neighborliness and improved bounds for other polynomial families in R^d

- **Def:** Finite $P \subseteq R^p$ is ***k*-neighborly** if every subset of k or less points of P determines a face of $\mathrm{conv}(P)$.
  Example: "neighborly in R^4" = "2-neighborly", while "1-neighborly"="convex position."
- **Thm [our work] (neighborliness for degree $\leq \boldsymbol{m}$ polynomials).** $\varphi: R^d \to R^{\binom{d+m}{m}-1}$: all monomials of degree $\leq m$.
  $S \subseteq R^d$: a finite set such that $\varphi(S)$ is in general linear position. Then $\varphi(S)$ is $\left(\binom{d+m/2}{m/2} - 1\right)$-neighborly.
  - Example d=4, m=2: embedding is 4-neighborly in R^{14}. Not neighborly (=7-neighborly) ⇒ no exact count of k-facets via our argument.
- **Thm [our work]:** $S \subseteq R^d$ is a set of n points in convex position ⇒ "# k-facets of S"$\leq \frac{n}{d}$ "max # k-facets in R^{d-1} for $n-1$ points."
  Proof idea: stereographic projection.
  Example: Best known bound for halving facets in R^3 is $O(n^{5/2})$, in R^4 is $O(n^{4-2/45})$. Our thm gives $O(n^{7/2})$ for points in convex position in R^4
- **Thm [our work]:** Like last theorem but assuming m-neighborly and giving better bound.
- Can use general upper bounds on # of halving facets in $R^{\binom{d+m}{m}-1}$ to get upper bounds on # of halving polynomials of degree $\leq m$.
- Thms above can be combined to get better than general bounds on # of halving polynomials of degree $\leq m$.
- Similar results for homogeneous polynomials.

Limits of neighborliness argument: neighborly embeddings

- Our mapping $\varphi: R^d \to R^p$ induces a d-manifold in R^p.
- **Def (k-neighborly embedding of a manifold)**: A d-manifold M embedded into R^p is k-neighborly if for every k-subset $S \subseteq M$ there is a hyperplane H that contains S and the rest of M is on one open side of H. Also neighborly=$\lfloor p/2 \rfloor$-neighborly.
  Examples: The moment curve $x \mapsto (x, x^2, \ldots, x^p)$ is neighborly.
  Image of map $(x, y) \mapsto (x^2, xy, y^2, x, y)$ is 1-neighborly. (Proof: a tiny empty circle passing through point)
  Not 2-neighborly. (Proof: 3 collinear points + other points)
- Problem of whether a map embeds points in neighborly position relates to a problem of Micha Perles:
  Open problem [Perles]: What is the smallest dimension $p(k, d)$ of the ambient space in which a k-neighborly d-dimensional manifold exists?
  - [Kalai Wigderson] $k(d+1) \leq p(k, d) \leq 2k(k-1)d$
- **Def (generally k-neighborly manifold)**: $\{n\text{−tuples of points that are } k\text{−neighborly}\} \subseteq R^{np}$ contains an open and dense set $\forall n$.
  Example: Image of map $(x, y) \mapsto (x^2, xy, y^2, x, y)$ is generally 2-neighborly (our result).
- In our case:
  Open problem [our work]: What is the smallest dimension $p_g(k, d)$ of the ambient space in which a generally k-neighborly d-dimensional manifold exists?
- **Our conjecture**: $p_g(k, d) \ = \ 2k + d - 1.$

Limits of neighborliness argument: neighborly embeddings

- In our case:
  Open problem [our work]: What is the smallest dimension $p_g(k,d)$ of the ambient space in which a generally k-neighborly d-dimensional manifold exists?
- **Our conjecture**: $p_g(k,d) = 2k + d - 1.$
- **Our results**:
  - $p_g(k,d) \leq 2k + d - 1.$
  - If "manifold" replaced by "algebraic variety," then $p_g(k,d) = 2k + d - 1.$
  - If our conjecture is true then exact counting via embeddings is not possible in R^d for $d \geq 3.$
    Proof for $d = 3$: R^3 embeds into R^p with $p \geq 2k + 2$, which implies $k < \lfloor p/2 \rfloor.$
    Same argument for larger $d.$

Expected number of k-facets

- Suppose S is an iid random sample of n points according to some distribution P in R^d.
  What is $E_P(n) \ = \ E(\text{\# of halving facets of } S)$?
  What is $E_P(k,n) \ = \ E(\text{\# of k}-\text{facets of } S)$?
- Assume measure via P of every hyperplane is 0. Implies general position of S a.s.
- [Bárány Steiger]
  - $E_P(n) = O(n^{d-1})$ if P is spherically symmetric.
  - $E_P(n) \ = \ O(n)$ if P is uniform in a convex body in R^2.
- [Clarkson] $E_P(n) \ = \ O(n^{d-1})$ if P is coordinate-wise independent.
- This is some evidence for belief that lower bound is closer to truth in k-set problem.
- What about more general distributions?
  - Note that there exist P in R^2 such that $E_P(n) \ = \ ne^{\Omega\left(\sqrt{\log n}\right)}$ (i.e. same as best lower bound for non-random).
  - It is believed growth of $E_P(n)$ can be as fast as deterministic case, but this is open.

Expected number of k-facets. Our results.

- **Thm:** If μ is a probability distribution on R^d such that the measure of every hyperplane is 0, then $E_\mu(n) = O(n^{d-1/2})$ (compare with best known deterministic "$O(n^{d-d^{-d}})$").

  Proof idea (follows from idea of [Bárány Steiger]): In R^2 for points $X_1, \ldots, X_n$ according to μ,

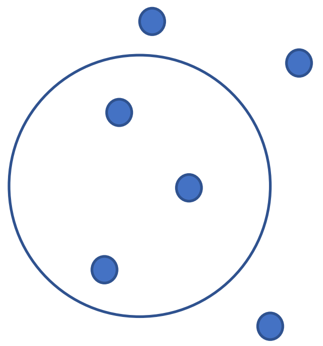
$$
\begin{aligned}
&E(\#\text{ of halving edges}) \\
&= \binom{n}{2} P(X_{n-1}, X_n \text{ is a halving line}) \\
&= \binom{n}{2}\binom{n-2}{(n-2)/2} P\big(X_1, \ldots, X_{(n-2)/2} \text{ below aff}(X_{n-1}, X_n), \text{rest above}\big) \\
&= \binom{n}{2}\binom{n-2}{(n-2)/2} \int \underbrace{\mu\big(\text{below aff}(x_{n-1}, x_n)\big)^{\frac{n-2}{2}} \Big(1-\mu\big(\text{below aff}(x_{n-1}, x_n)\big)\Big)^{\frac{n-2}{2}}}_{\text{“}t(1-t) \le 1/4\text{”}} \, d\mu(x_{n-1}) d\mu(x_n) \\
&\le \frac{\binom{n}{2}\binom{n-2}{(n-2)/2}}{4^{(n-2)/2}} \\
&= O\big(n^{3/2}\big).
\end{aligned}
$$

Expected number of k-facets. Our results.

- **Thm:** For any probability P on R^2 such that the measure of every line is 0, $E_P(k,n) \leq 10n(k+1)^{1/4}$
  (compare with best known deterministic, $O(nk^{1/3})$ [Dey]).
  Proof idea:
  - Use vertical lines equipartition on P (not on the points).
  - Use argument in previous theorem to bound k-edges within parts.
  - Use convex chains to bound k-edges across parts.
- An idea: replace vertical lines partitioning by polynomial partitioning.
  Open problem: What is the maximum (finite) # of times that an irreducible non-singular degree r algebraic curve can intersect the k-edge graph of a set of n points in the plane?
  We prove between nr and nr^2. If true is nr, we can improve our bound on $E_P(k,n)$.

Expected number of k-facets. Our results.

- The argument we use from [Bárány Steiger] gives morally $O\left(n^{(\#\text{d.o.f. of shapes})-1/2}\right)$ bound for k-sets. How loose is this?
- **[Our result]** If we allow shapes beyond hyperplanes/halfspaces, it is tight:
  For certain distributions on R^2 and translations of any fixed strictly convex curve (two d.o.f.),
  $E(\#\text{ of induced k}-\text{sets on } n \text{ random points}) = \Theta(n^{3/2})$
  (up to polylog factors).



Expected number of k-facets of random Gaussian point sets in R^d

- $S = \{X_1, \ldots, X_n\}$: iid random Gaussian sample of n points in R^d. $P = \text{conv}(S)$.
- $E(\# \text{ of facets of } P)$?
  - For fixed d and as $n \to \infty$, well studied. [Raynaud] [Rényi Sulanke] give precise asymptotics.
  - Both d and n grow? Particularly, proportional regime $n \approx cd$ (relevant for applications).
    - [Vershik Sporyshev] [Donoho Tanner] In proportional regime: degree of neighborliness of P and indirect information about # of facets of P.
    - [Böröczky Lugosi Reitzner] $n \geq e^e d$ or $n - d = o(d)$. In proportional regime, provides exponential upper and lower bounds.

Expected number of k-facets of random Gaussian point sets in R^d: our results.

- $S = \{X_1, \dots, X_n\}$: iid random Gaussian sample of n points in R^d. $P = \mathrm{conv}(S)$.
- **Thm [our work]**: If $n/d \to \alpha > 1$ and $k/(n-d) \to r \in [0,1]$, then $E(\#\text{ of k}-\text{facets of } S) = C^{d+o(d)}$ with an easy way to determine C.
- Example: If $n = 2d$ and $k = 0$ (facets), then $C = 4\sqrt{2\pi}\max_{y\in R} \Phi(y)\Phi'(y) \approx 2.44$, where Φ is CDF of $N(0,1)$.
- Proof idea:
  - Extend formula in [Hug Munsonius Reitzner] from facets to k-facets to get equivalent 1-dim problem:
    Thm [our work]: $P(X_1, \dots, X_d \text{ is a k}-\text{facet of } S) = P(Y \text{ is k+1}^{\text{st}} \text{ largest or k+1}^{\text{st}} \text{ smallest in } \{Y, Y_1, \dots, Y_{n-d}\})$, where $Y \sim N(0,1/d)$ and $Y_i \sim N(0,1)$ and all are independent.
    Proof idea: Project onto line perpendicular to $\mathrm{aff}(X_1, \dots, X_d)$ and determine distributions of projected random variables.
  - Write then $P(X_1, \dots, X_d \text{ is a k}-\text{facet of } S) = 2\binom{n-d}{k}\frac{\sqrt{d}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \Phi(y)^k\left(1-\Phi(y)\right)^{n-d-k} e^{-\frac{dy^2}{2}}dy$
  - E.g. for $n = 2d$, $k = 0$ (facet): $P(X_1, \dots, X_d \text{ is a facet of } S) = 2\frac{\sqrt{d}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(1-\Phi(y))^d e^{-\frac{dy^2}{2}}dy$
  - Use the following "easy" asymptotic expansion of integral: $\int_{R^d} f(x)^p\, dx = \|f\|_\infty^{p+o(p)}$ as $p \to \infty$.
    Proof: Start with "L^p norm of function converges to L^∞ norm as $p \to \infty$" under mild assumptions, then raise to pth power.