Anti-concentration and the Gap-Hamming problem

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Technion

Outline

Anti-concentration

1. The Littlewood-Offord problem
2. Erdos’s answer
3. Halasz’s approach

Communication complexity

1. Rectangle partitions
2. Connection to streaming algorithms
3. Gap-Hamming

Our work

Halasz generalized to rectangles

Littlewood-Offord Problem

Suppose

$x \in \mathbb{R}^n$ has no zero coordinates.

$Y \in \{\pm 1\}^n$ is uniformly random.

$\langle x, Y \rangle = \sum_{i=1}^{n} x_i Y_i$.

Q: What is $\max_k \Pr[\langle x, Y \rangle = k]$?

Littlewood-Offord Problem

Suppose
$x \in \mathbb{R}^n$ has no zero coordinates.
$Y \in \{\pm 1\}^n$ is uniformly random.
$\langle x, Y \rangle = \sum_{i=1}^{n} x_i Y_i$.
Q: What is $\max_k \Pr[\langle x, Y \rangle = k]$?

Example:
$x = (1, 3, 3^2, \ldots, 3^{n-1})$

Then $\langle x, Y \rangle$ determines Y, so $\max_k \Pr[\langle x, Y \rangle = k] = 2^{-n}$.

Littlewood-Offord Problem

Suppose
$x \in \mathbb{R}^n$ has no zero coordinates.
$Y \in \{\pm 1\}^n$ is uniformly random.
$\langle x, Y \rangle = \sum_{i=1}^{n} x_i Y_i$.
Q: What is $\max_k \Pr[\langle x, Y \rangle = k]$?

Example:

$x = (1,1,\ldots,1)$
Let W denote number of coords of Y that are -1.
$\langle x, Y \rangle = k$ means $(n - W) - W = k$, so $W = (n-k)/2$.

$$\Pr[\langle x, Y \rangle = k] = \binom{n}{(n-k)/2} \cdot 2^{-n}.$$

$$\max_k \Pr[\langle x, Y \rangle = k] = \binom{n}{\lfloor n/2 \rfloor} \cdot 2^{-n} = O(1/\sqrt{n}).$$

Littlewood-Offord Problem

Suppose
$x \in \mathbb{R}^n$ has no zero coordinates.
$Y \in \{\pm 1\}^n$ is uniformly random.
Q: What is $\max_k \Pr[\langle x, Y \rangle = k]$?

Erdos:

wlog $x_i > 0$.

Claim: For each k, $S = \{y : \langle x, y \rangle = k\} \subseteq \{\pm 1\}^n$ is an *antichain*.

Antichain: $y, y' \in S$ means cannot have $y_i > y_i'$ for all i.

Sperner's theorem: The largest antichain in $\{\pm 1\}^n$ has size $\binom{n}{\lfloor n/2 \rfloor}$.

Thm: $\max_k \Pr[\langle x, Y \rangle = k] \leq \binom{n}{\lfloor n/2 \rfloor} \cdot 2^{-n} \approx 1/\sqrt{n}$.

Littlewood-Offord Problem

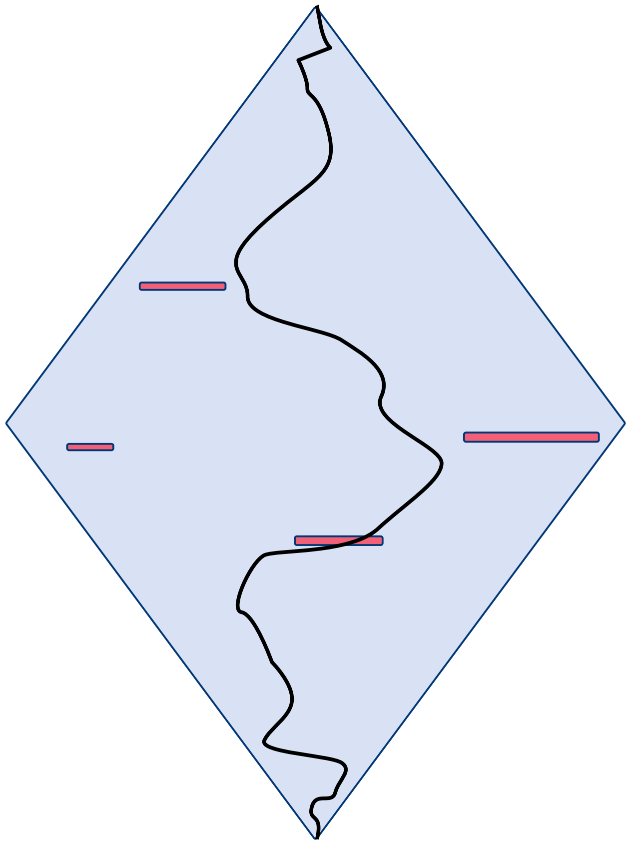
Suppose
$x \in \mathbb{R}^n$ has no zero coordinates.
$Y \in \{\pm 1\}^n$ is uniformly random.
Q: What is $\max_k \Pr[\langle x, Y \rangle = k]$?

Sperner's theorem: The largest antichain in $\{\pm 1\}^n$ has size $\binom{n}{\lfloor n/2 \rfloor}$.

Pf: Let $A \subseteq \{\pm 1\}^n$ be an antichain. Let a_i be number of elements of A with i 1's.

Let

$(-1,\ldots,-1) = Y^{(0)}, Y^{(1)}, \ldots, Y^{(n)} = (1,\ldots,1)$ be a random chain,
Where $Y^{(i)}$ is obtained from $Y^{(i-1)}$ by flipping random -1 to 1.



$$1 \geq \text{probability that the chain passes through } A$$
$$\geq \frac{a_0}{\binom{n}{0}} + \frac{a_1}{\binom{n}{1}} + \ldots + \frac{a_n}{\binom{n}{n}} \geq \frac{|A|}{\binom{n}{\lfloor n/2 \rfloor}}.$$

Littlewood-Offord Problem

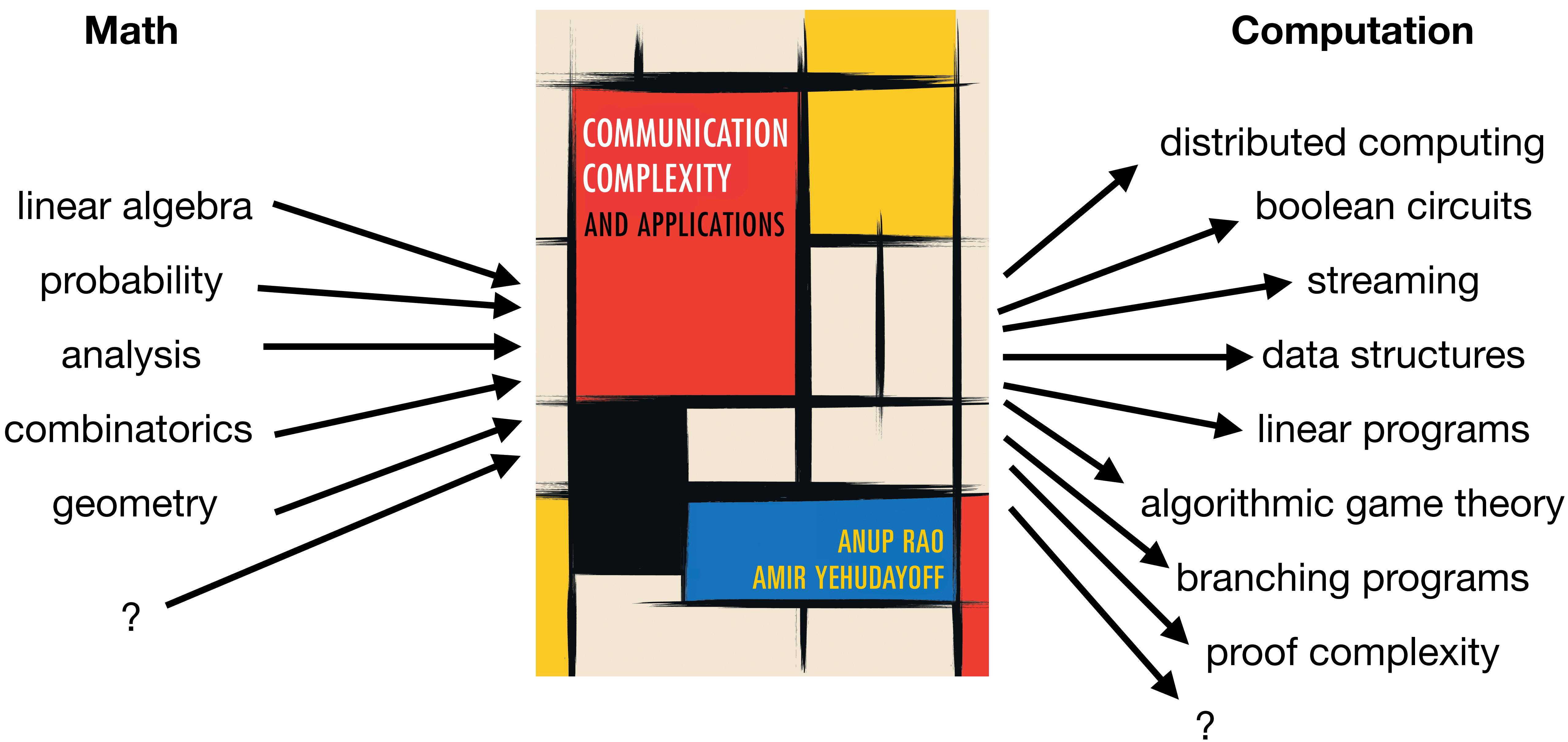
$x \in \mathbb{R}^n$ has no zero coordinates.
$Y \in \{\pm 1\}^n$ is uniformly random.

Erdös: $\max_k \Pr[\langle x, Y \rangle = k] \leq \binom{n}{\lfloor n/2 \rfloor} \cdot 2^{-n} \leq O(1/\sqrt{n})$.

Erdös-Moser, Sárkozy-Szemerédi: If coordinates x_i distinct,
$\Pr[\langle x, Y \rangle = k] \leq O(n^{-3/2})$.

Halász: Suppose $x \in \mathbb{Z}^n$, let $0 \leq \theta \leq 1$ be uniform.

$$\begin{aligned}
\Pr[\langle x, Y \rangle = k] &= \mathbb{E}_{\theta,Y}[\exp(2\pi i \cdot \theta \cdot (\langle x, Y \rangle - k))] \\
&\leq \mathbb{E}_\theta | \mathbb{E}_Y[\exp(2\pi i \cdot \theta \cdot \langle x, Y \rangle)] | \\
&= \mathbb{E}_\theta \left| \mathbb{E}_Y \left[\prod_{j=1}^{n} \exp(2\pi i \cdot \theta \cdot x_j Y_j) \right] \right| \\
&= \mathbb{E}_\theta \left| \prod_{j=1}^{n} \cos(2\pi \theta x_j) \right| \leq O(1/\sqrt{n}).
\end{aligned}$$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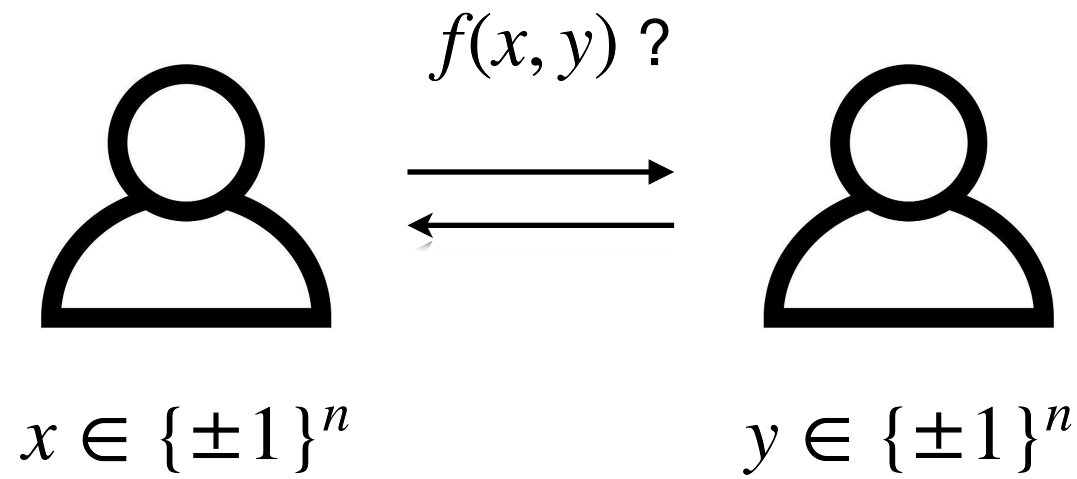
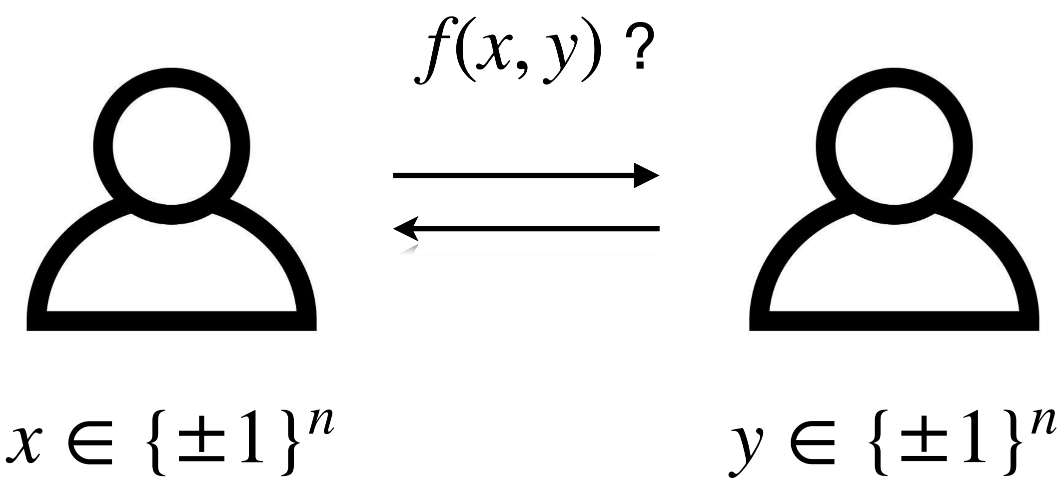
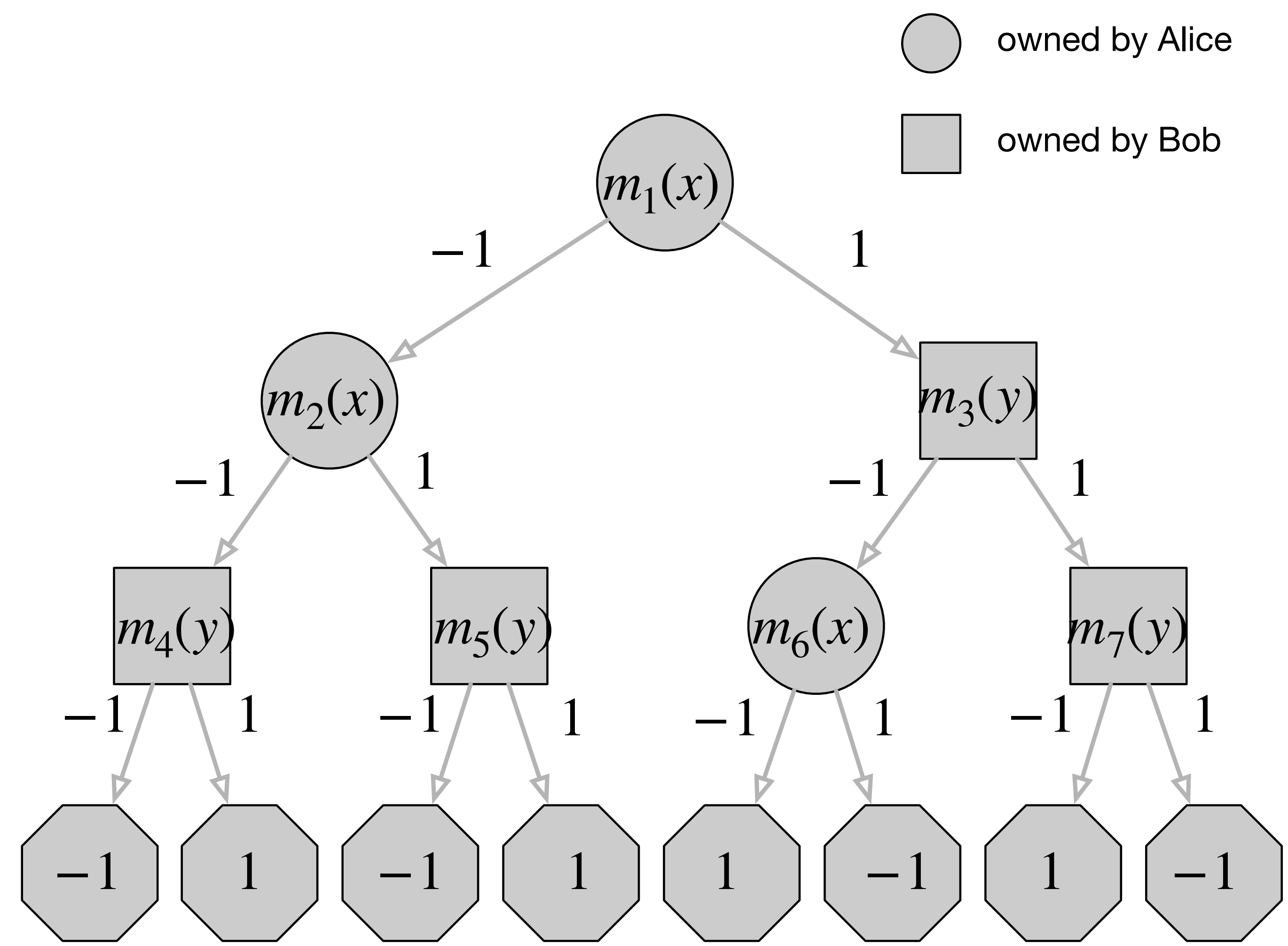
Communication complexity



How long does their conversation need to be?

Communication complexity





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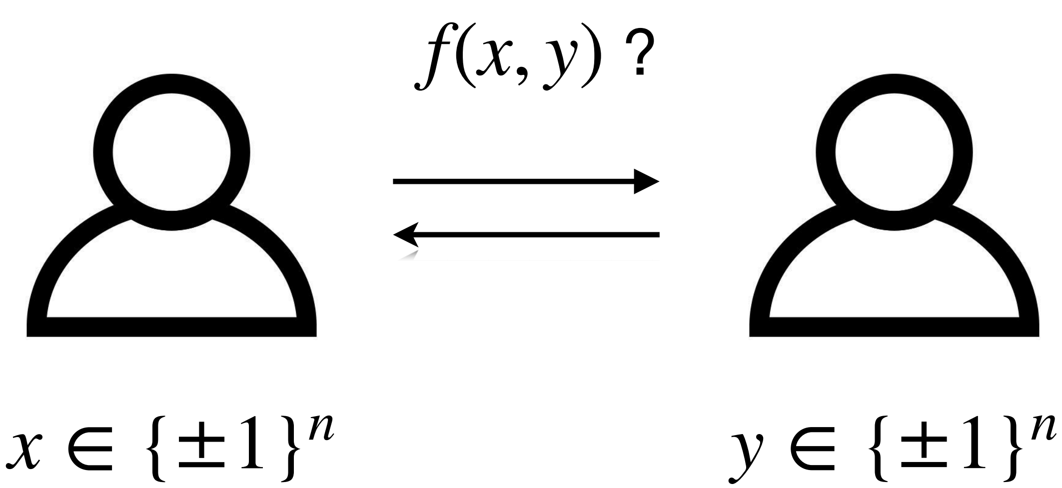
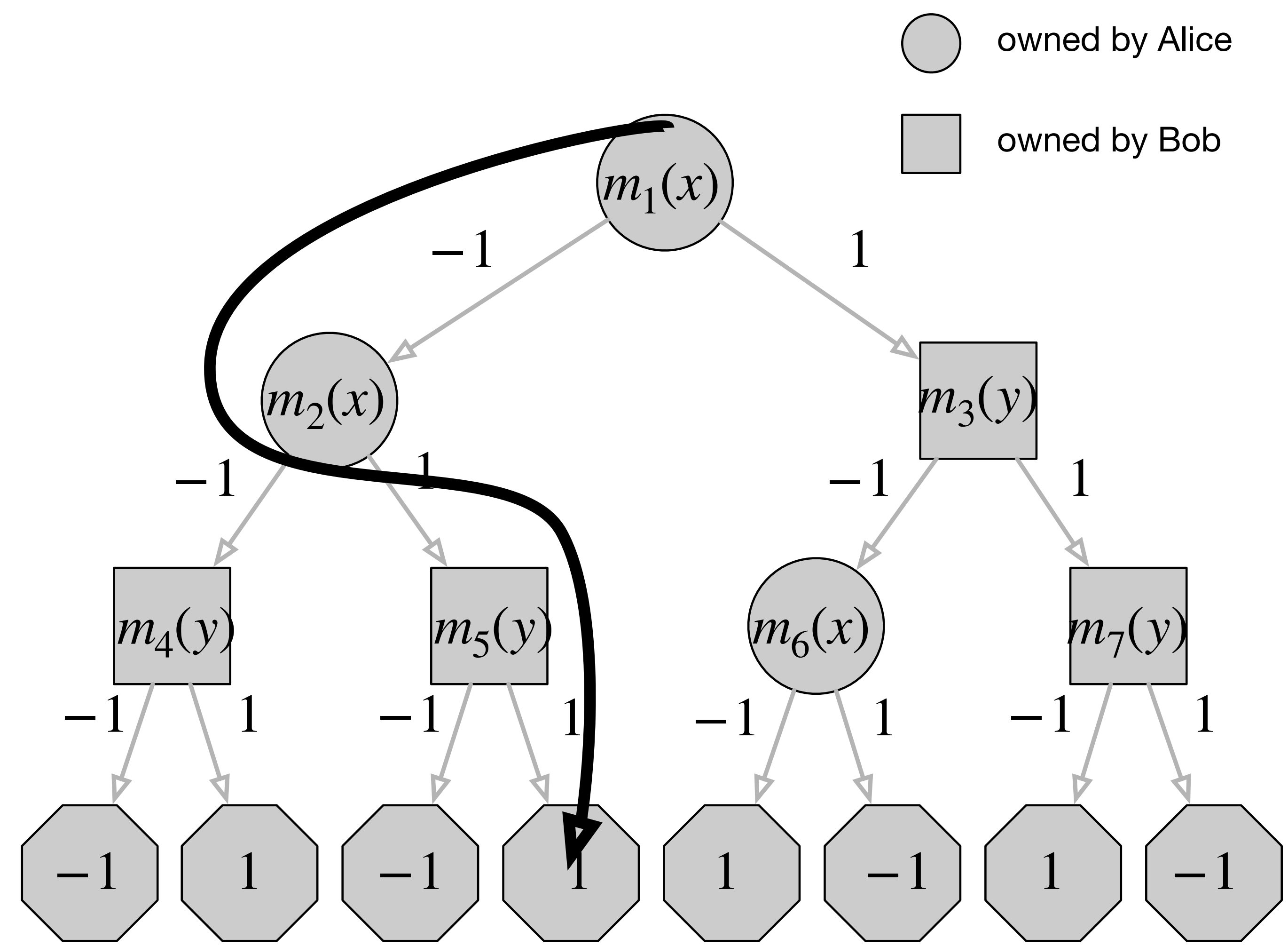
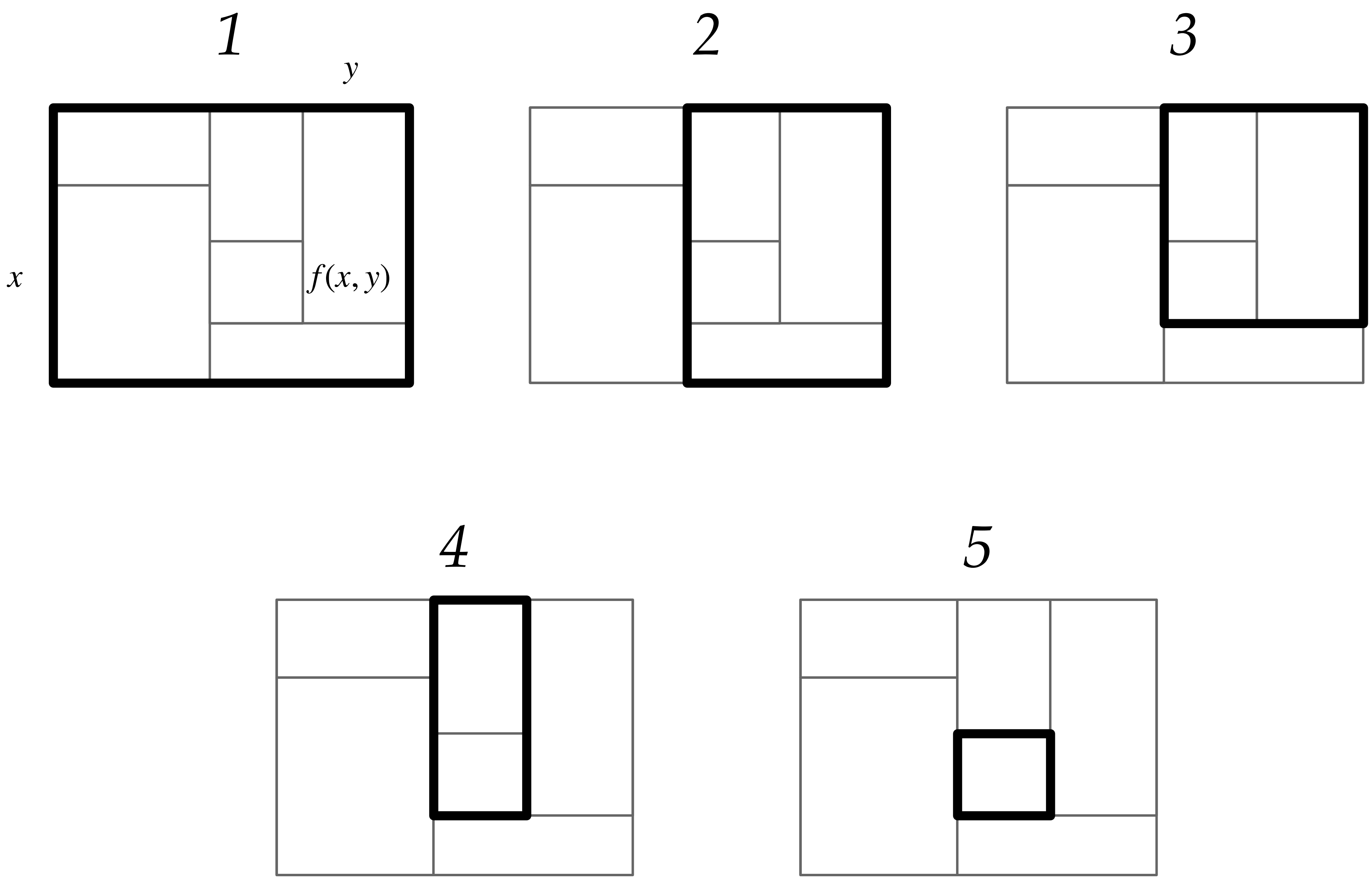
Communication complexity





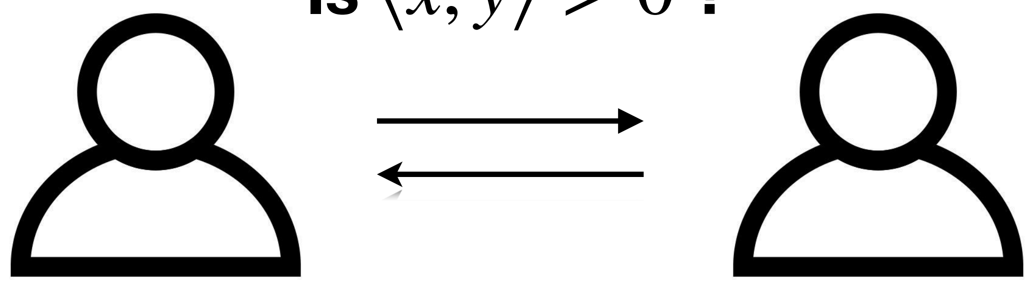
How long does their conversation need to be?

Small communication = partition into few rectangles



The Gap-Hamming Problem

Is $\langle x, y \rangle > 0$ **?**



$x \in \{\pm 1\}^n$ $y \in \{\pm 1\}^n$

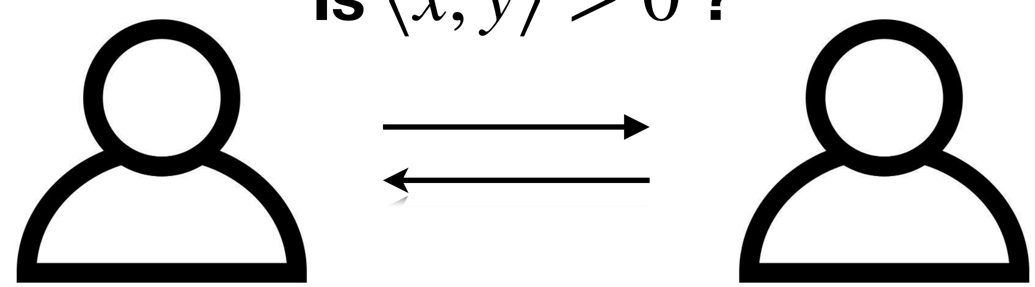
promise

$|\langle x, y \rangle| \geq \sqrt{n}$

How long does their conversation need to be?

Exact Gap-Hamming Problem

Is $\langle x, y \rangle > 0$ **?**



$x \in \{\pm 1\}^n$ $\quad$ $y \in \{\pm 1\}^n$

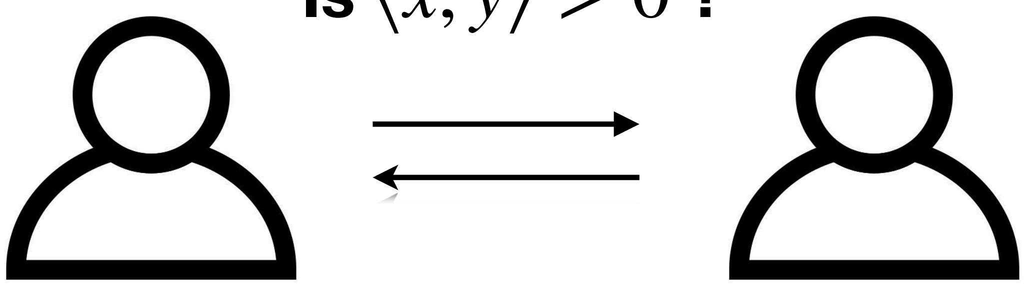
promise

$$|\langle x, y \rangle| = \sqrt{n}$$

How long does their conversation need to be?

Gap-Hamming Problem

Is $\langle x, y \rangle > 0$ **?**



$x \in \{\pm 1\}^n$ $y \in \{\pm 1\}^n$

promise

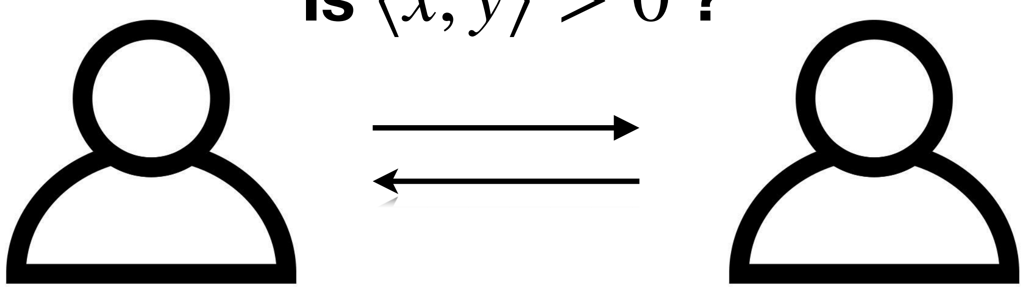
$|\langle x, y \rangle| \geq \sqrt{n}$

History:

1. Posed by [**Indyk-Woodruff**], motivated by streaming.
2. $\Omega(n)$ communication required [**Chakrabarti-Regev**]: cube -> Gaussians
3. [**Sherstov**]: SVD, Talagrand's inequality
4. [**Vidick**]: cube -> Gaussians
5. [**Our work**]: Fourier analysis

Exact Gap-Hamming Problem

Is $\langle x, y \rangle > 0$ **?**



$x \in \{\pm 1\}^n$ $y \in \{\pm 1\}^n$

promise

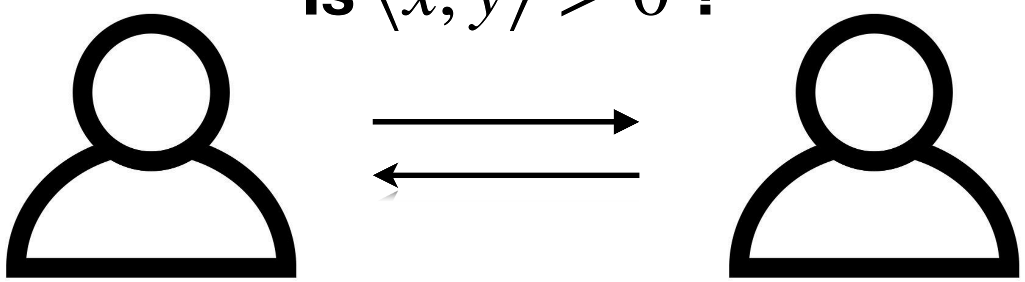
$|\langle x, y \rangle| = \sqrt{n}$

Observation:	*flipping* x_1 *changes*
n **and** $\prod_{j=1}^{n} x_j y_j$	sign
determine	
$\langle x, y \rangle \mod 4$	$+2 \mod 4$

[**Chakrabarti-Regev**]: What is the communication required in general?

Exact Gap-Hamming Problem

Is $\langle x, y \rangle > 0$ **?**



$x \in \{\pm 1\}^n$ $y \in \{\pm 1\}^n$

promise

$|\langle x, y \rangle| = \sqrt{n}$

New in our work:

$\Omega(n)$ communication is required
in general
(Fourier analysis)

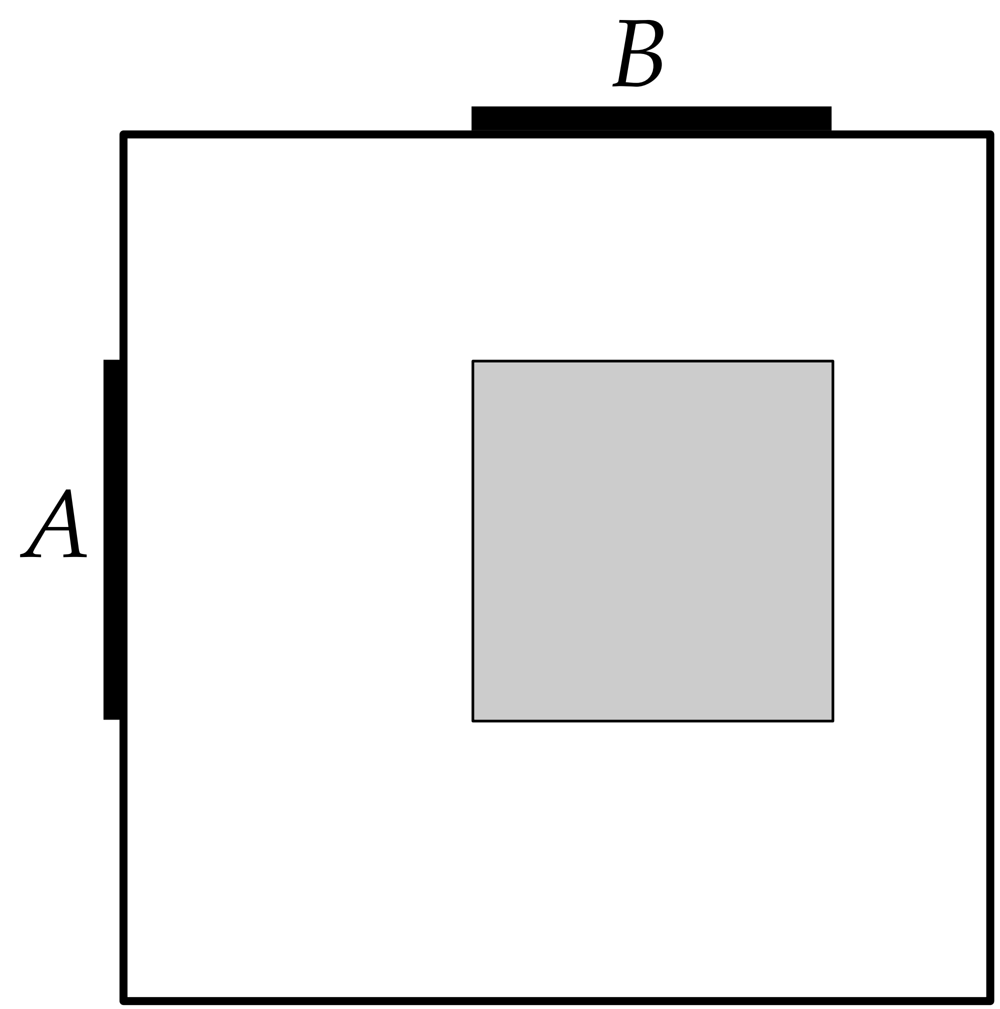
Large Rectangles

$X \in_R A \subseteq \{\pm 1\}^n$
$Y \in_R B \subseteq \{\pm 1\}^n$

What can we say about the distribution of $\langle X, Y \rangle$ if $|A| \cdot |B|$ is large?

[Chakrabarti-Regev], [Sherstov], [Vidick]:
If $|A| \cdot |B| > 2^{1.99n}$,
$\Pr[\,|\langle X, Y \rangle| \leq \sqrt{n}/100] \leq 0.99.$

Anti-concentration

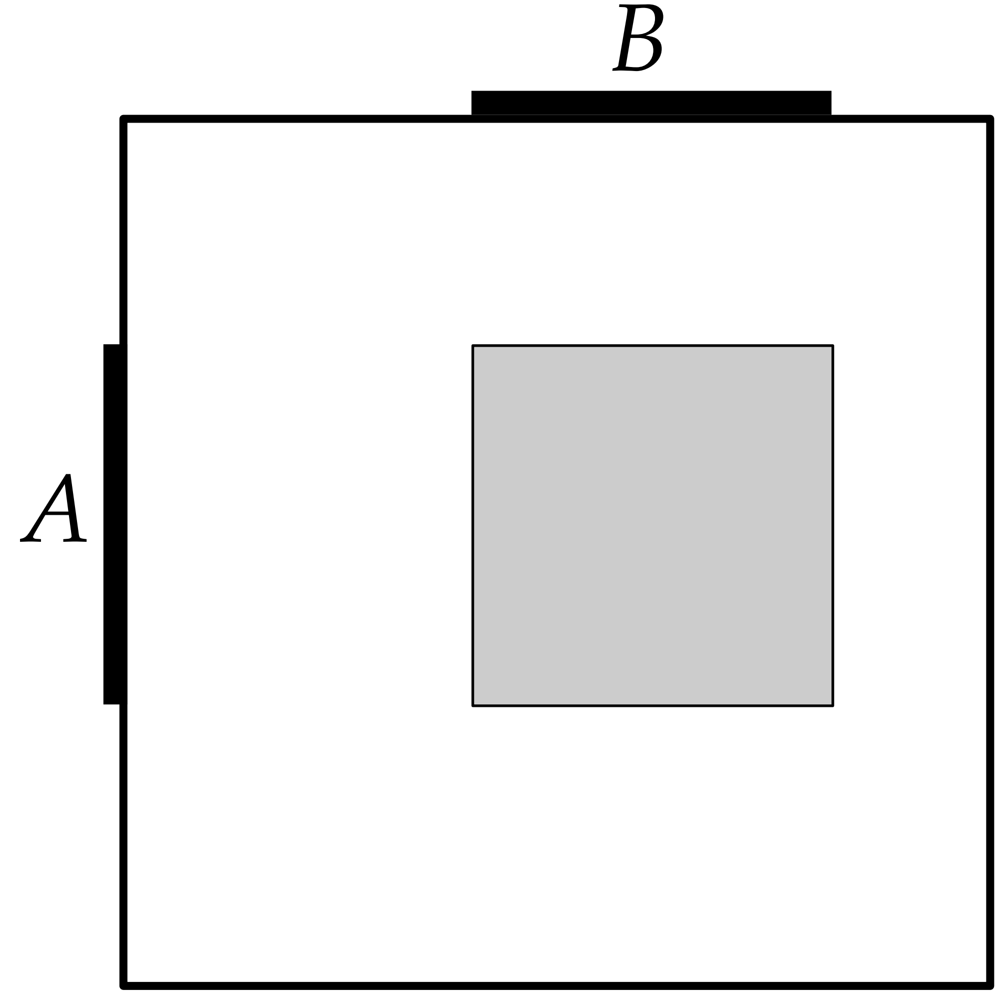


(In our work: optimal anti-concentration)

$X \in_R A \subseteq \{\pm 1\}^n$

$Y \in_R B \subseteq \{\pm 1\}^n$

What can we say about the distribution of $\langle X, Y \rangle$ if $|A| \cdot |B|$ is large?



Example 1:

$A = B = \{\pm 1\}^n$, n even, then $\Pr[\langle X, Y \rangle = 0] = \Theta(1/\sqrt{n})$.

Example 2:

sum to 0

A 11111111111111****************

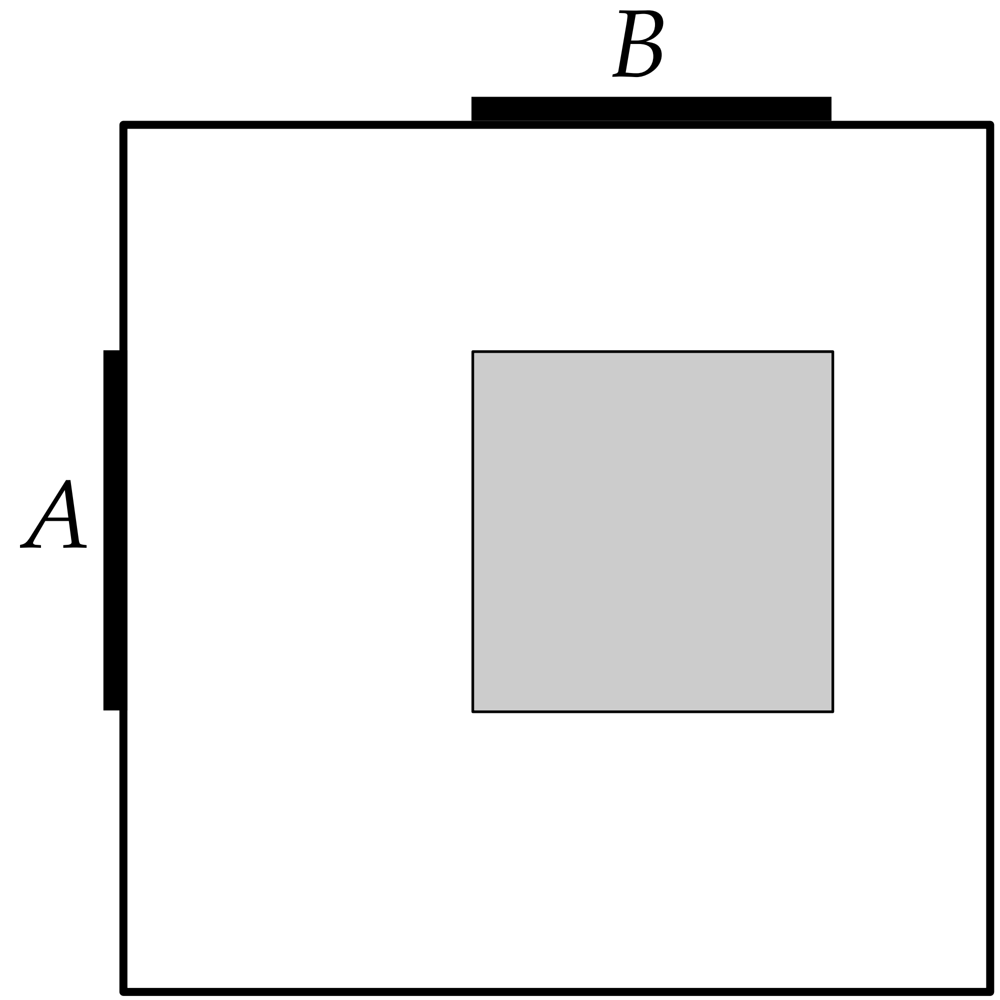
B ****************11111111111111

sum to 0

$|A| \cdot |B| \geq \Omega(2^n/n)$ $\langle X, Y \rangle = 0$

$$X \in_R A \subseteq \{\pm 1\}^n$$
$$Y \in_R B \subseteq \{\pm 1\}^n$$

What can we say about the distribution of $\langle X, Y \rangle$ if $|A| \cdot |B|$ is large?



Our Results

Thm 1: If $|A| \cdot |B| \geq 2^{1.01n}$, for all k, $\Pr[\langle X, Y \rangle = k] \leq O(1/\sqrt{n})$.

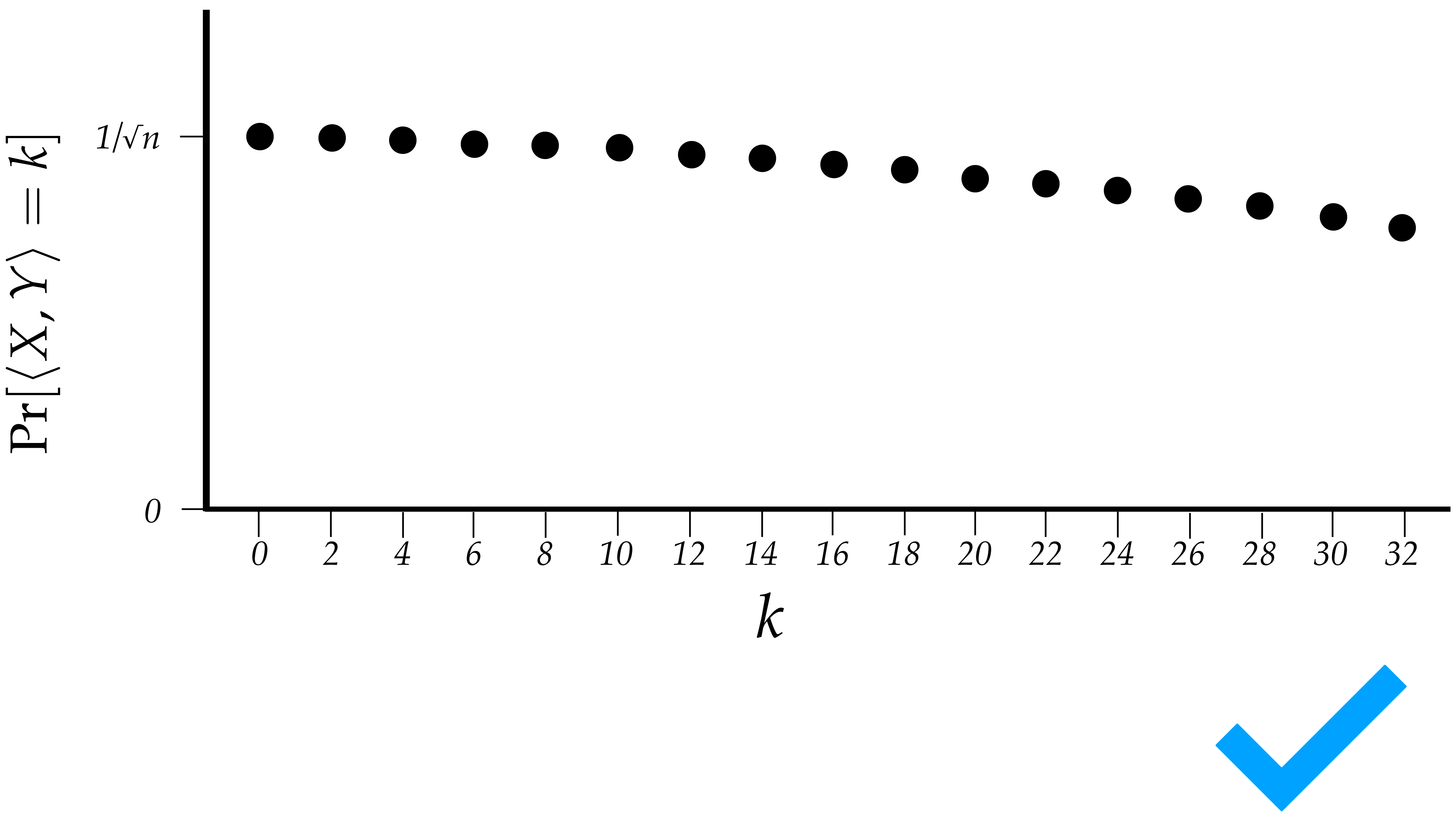
Remark: This implies all past results.

Thm 2: … except with exp small probability over $x \in A$, $\Pr[\langle x, Y \rangle = k] \leq O(1/\sqrt{n})$.

Thm 3: … $|\Pr[\langle x, Y \rangle = k] - \Pr[\langle x, Y \rangle = k + 4]| \leq O(1/n)$.

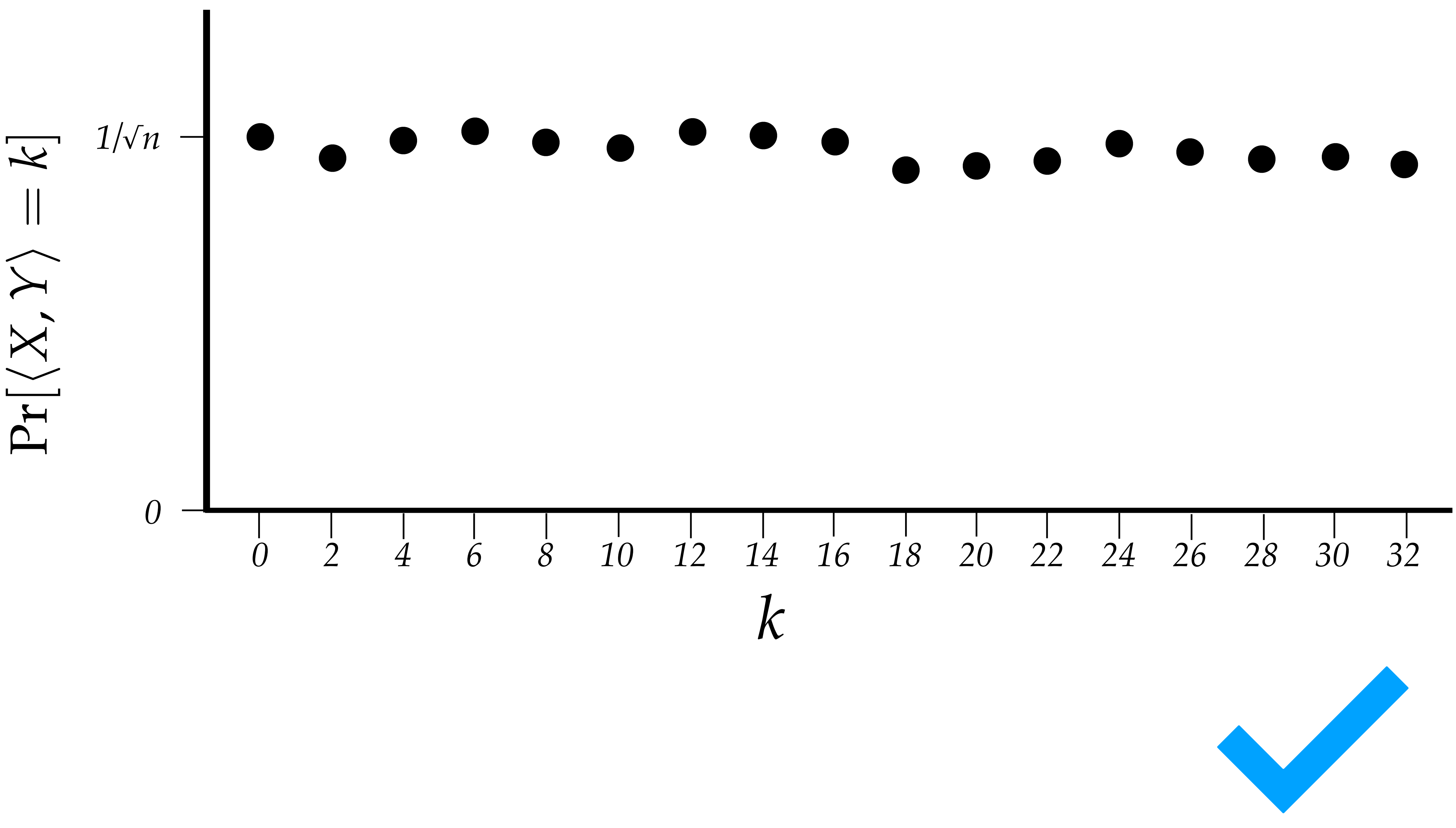
Remark: This is false if 4 is replaced with 6 or 2, as we saw.

Thm: For all k, if $|A| \cdot |B| > 2^{1.01n}$,
$$|\Pr[\langle X, Y \rangle = k] - \Pr[\langle X, Y \rangle = k+4]| \leq O(1/n) .$$



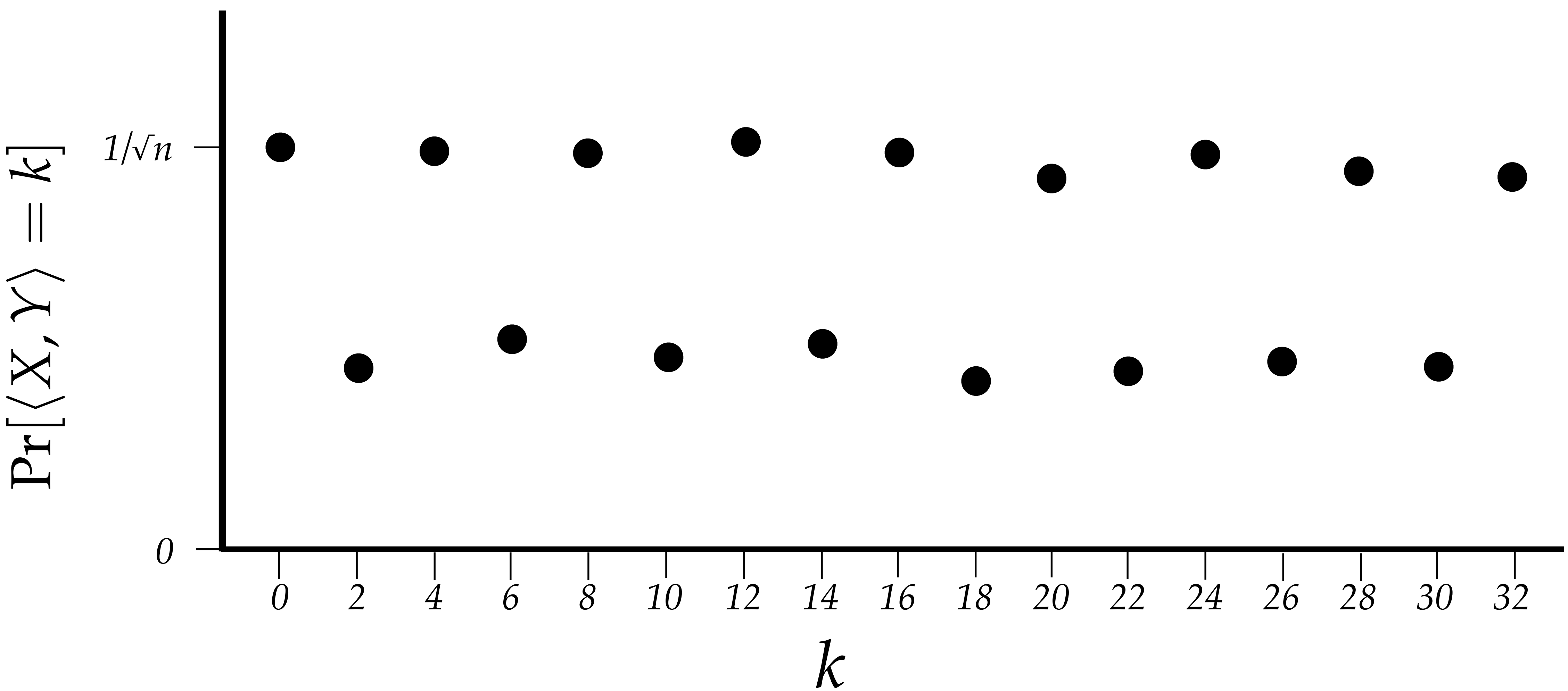
Thm: For all k, if $|A| \cdot |B| > 2^{1.01n}$,

$$|\Pr[\langle X, Y \rangle = k] - \Pr[\langle X, Y \rangle = k+4]| \leq O(1/n)\,.$$



Thm: For all k, if $|A| \cdot |B| > 2^{1.01n}$,

$$|\Pr[\langle X, Y \rangle = k] - \Pr[\langle X, Y \rangle = k + 4]| \leq O(1/n)\,.$$



Thm: For all k, if $|A| \cdot |B| > 2^{1.01n}$,
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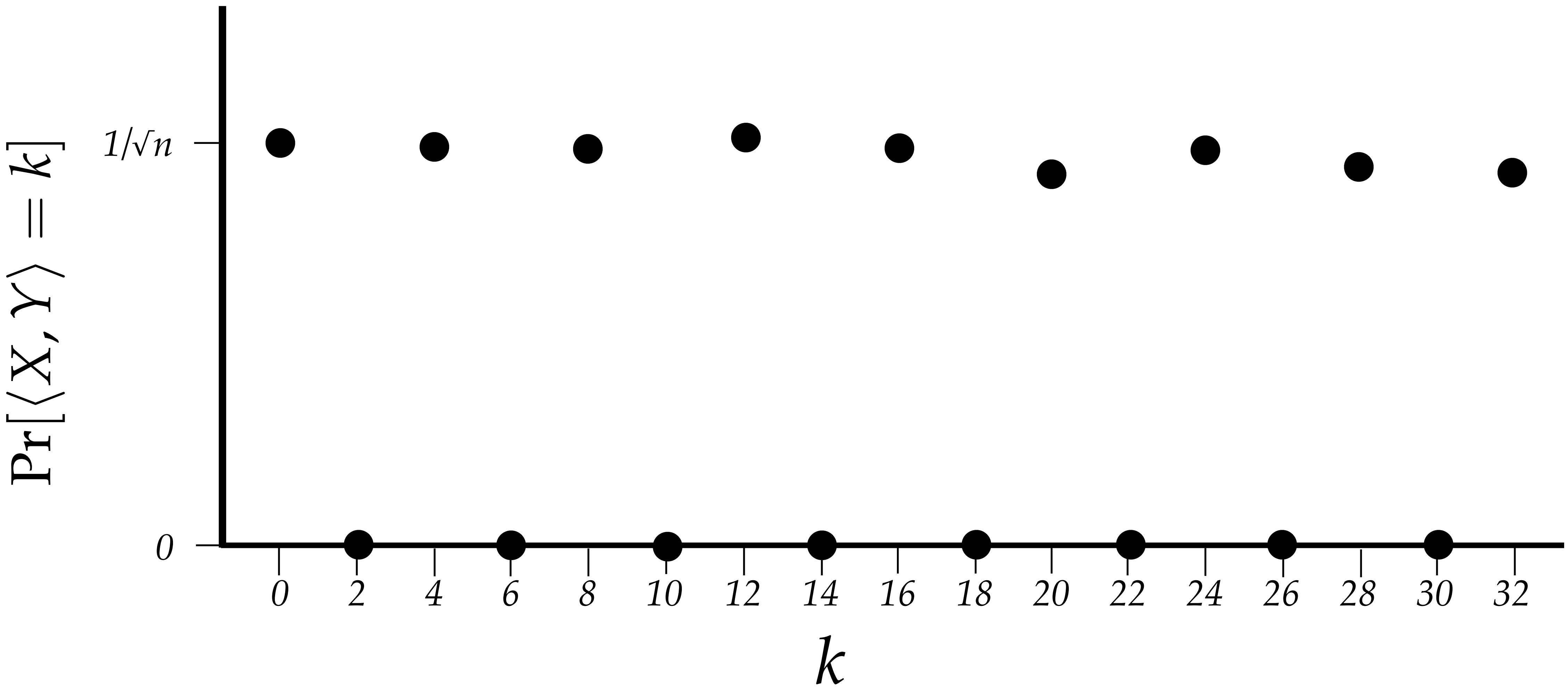
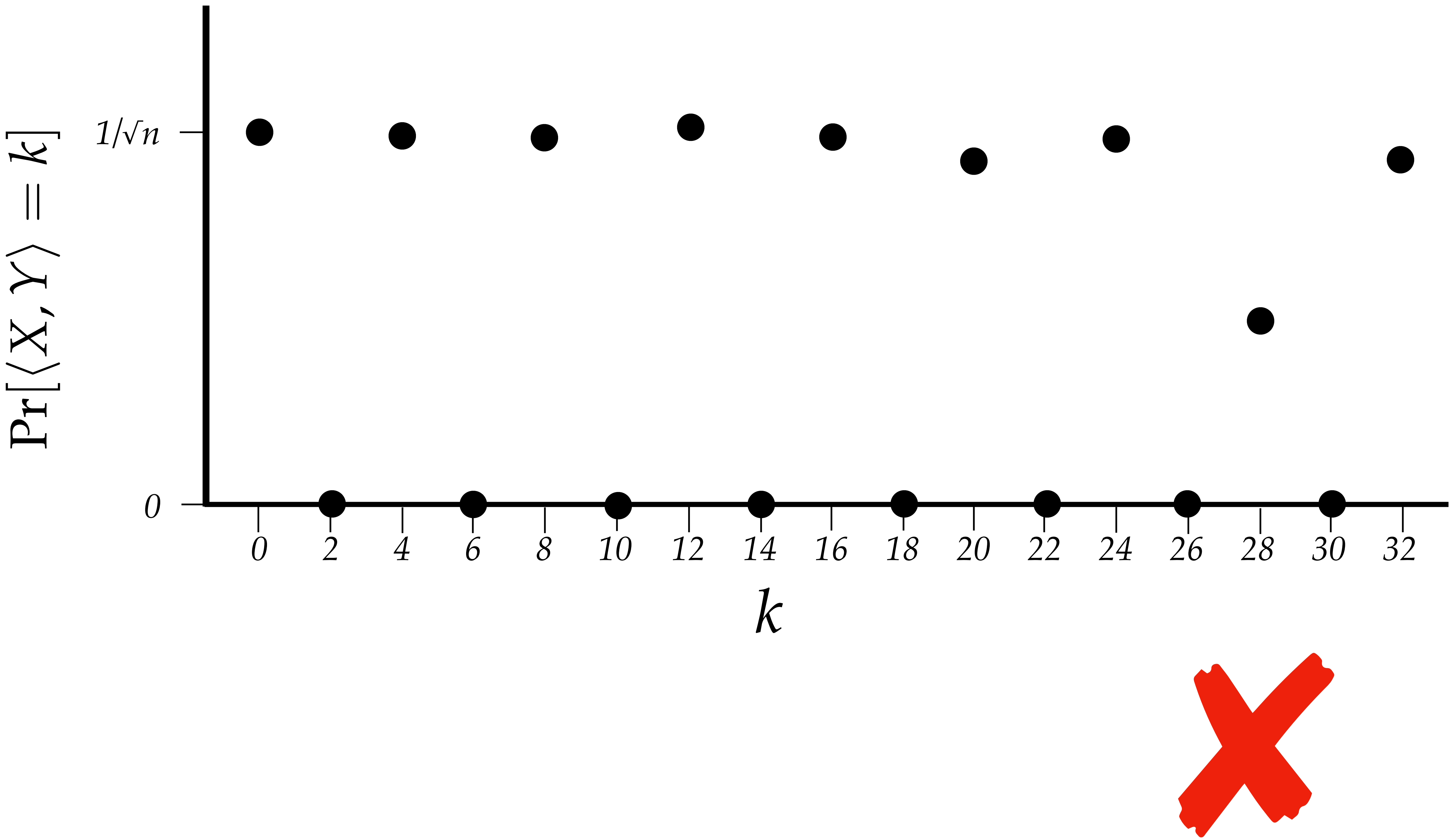


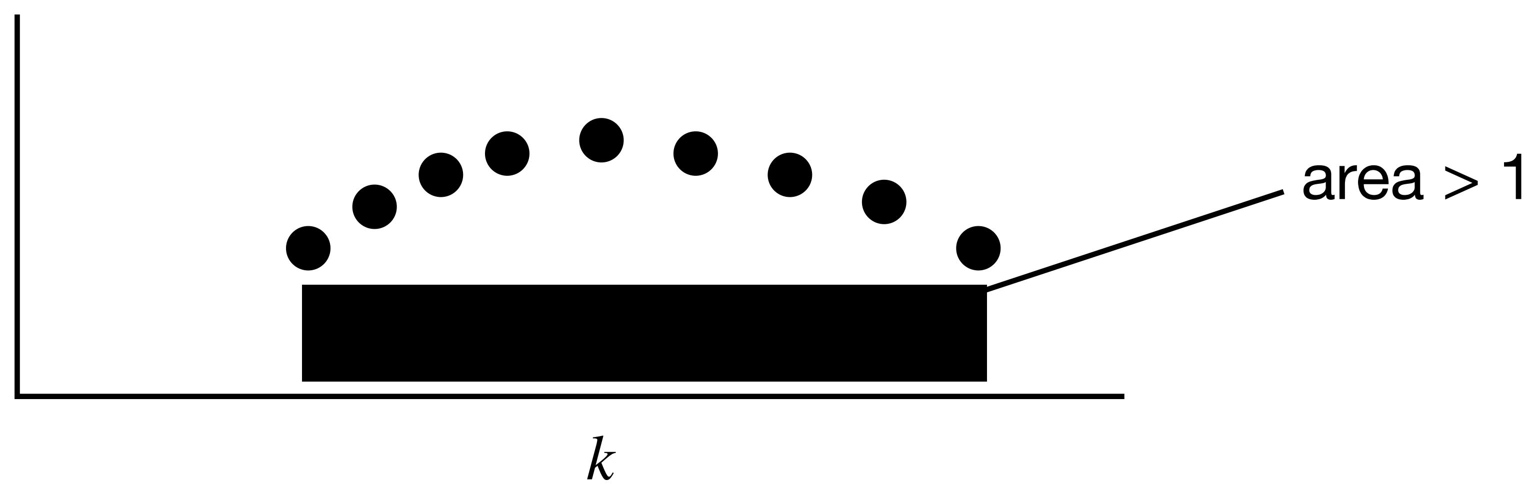
Thm: For all k, if $|A| \cdot |B| > 2^{1.01n}$,
$|\Pr[\langle X, Y \rangle = k] - \Pr[\langle X, Y \rangle = k + 4]| \leq O(1/n)$.



Thm: For all k, if $|A| \cdot |B| > 2^{1.01n}$,
$$|\Pr[\langle X, Y\rangle = k] - \Pr[\langle X, Y\rangle = k+4]| \leq O(1/n)\,.$$

Corollary 1: For all k, $\Pr[\langle X, Y\rangle = k] \leq O(1/\sqrt{n})\,.$

Pf Sketch:



Corollary 2: Exact-gap-Hamming requires $\Omega(n)$ communication.

Pf Sketch:

If not, sample a rectangle R by feeding inputs with $|\langle X', Y'\rangle| = \sqrt{n}$. Then whp,

- $|A| \cdot |B| \geq 2^{1.01n}$
- $\Pr[\,|\langle X, Y\rangle| = \sqrt{n}\,|\,R] \geq \Omega(1/\sqrt{n})$

So, the protocol must make an error with significant probability.

[Halász]: Fourier analytic approach

Suppose $Y \in \{\pm 1\}^n$ is uniform, $x \in \mathbb{Z}^n_{\neq 0}$

$$\Pr[\langle x, Y\rangle = k] = \mathbb{E}_{\theta,Y}[\exp(2\pi i \cdot \theta \cdot (\langle x, Y\rangle - k))]$$

$$\leq \mathbb{E}_{\theta} | \mathbb{E}_{Y}[\exp(2\pi i \cdot \theta \cdot \langle x, Y\rangle)] |$$

$$= \mathbb{E}_{\theta} \left| \mathbb{E}_{Y}\left[\prod_{j=1}^{n} \exp(2\pi i \cdot \theta \cdot x_j Y_j)\right] \right|$$

$$= \mathbb{E}_{\theta} \left| \prod_{j=1}^{n} \cos(2\pi\theta x_j) \right| \leq O(1/\sqrt{n}) .$$

Challenge:
In our setting the coordinates of Y are correlated!

Technical Thm: For all θ, if $|A| \cdot |B| = 2^{1.01n}$, then except with exp small probability over $x \in A$,
$$|\mathbb{E}_Y[\exp(2\pi i \cdot \theta \cdot \langle x, Y \rangle)]| < \exp(-\Omega(n \sin^2(4\pi\theta))) .$$

Thm: For all k, if $X \in A, Y \in B, |A| \cdot |B| > 2^{1.01n}$,
$$|\Pr[\langle X, Y \rangle = k] - \Pr[\langle X, Y \rangle = k+4]| \le O(1/n) .$$

Pf:

Using: $\Pr[\langle x, Y \rangle = k] = \mathbb{E}_{\theta,Y}[\exp(2\pi i \cdot \theta \cdot (\langle x, Y \rangle - k))]$,

$$|\Pr[\langle X, Y \rangle = k] - \Pr[\langle X, Y \rangle = k + 4]|$$

$$\lessapprox \mathbb{E}_\theta[|\exp(-2\pi i\theta \cdot (k+2))| \cdot |(\exp(4\pi i\theta) - \exp(-4\pi i\theta)) \cdot \exp(-\Omega(n \sin^2(4\pi\theta)))|]$$

$$\le 2 \cdot \mathbb{E}_\theta[|\sin(4\pi\theta) \cdot \exp(-\Omega(n \sin^2(4\pi\theta)))|]$$

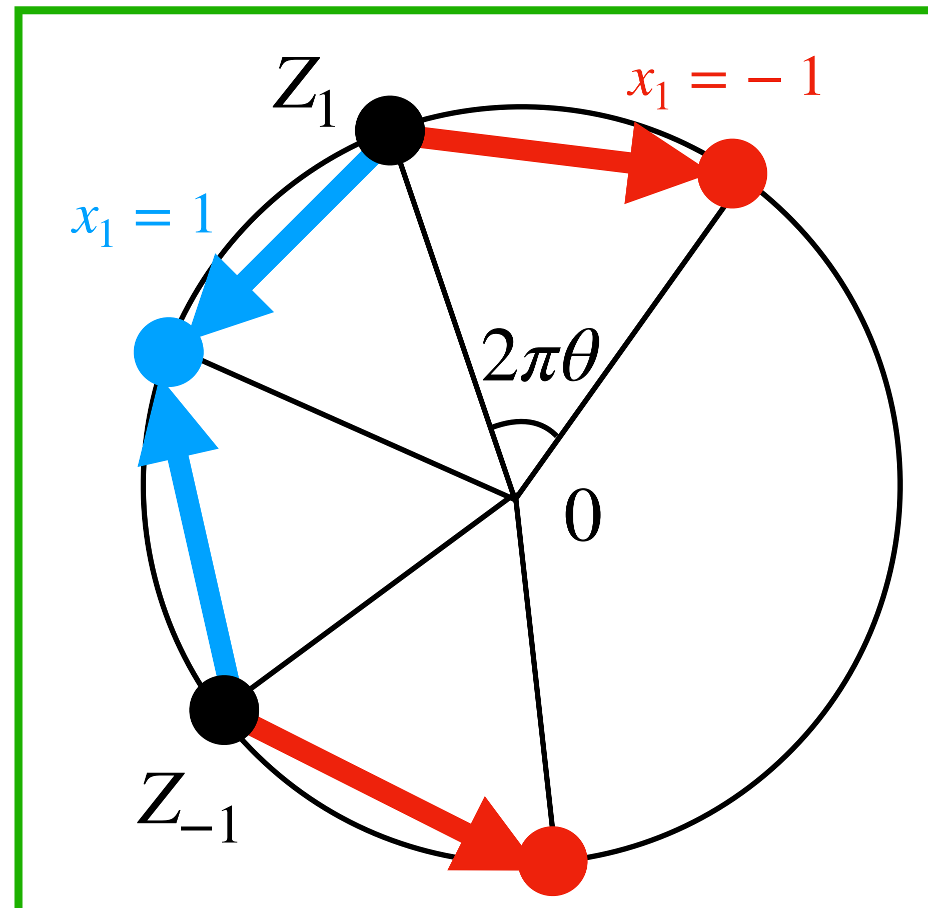
$$\le O(1/n) .$$

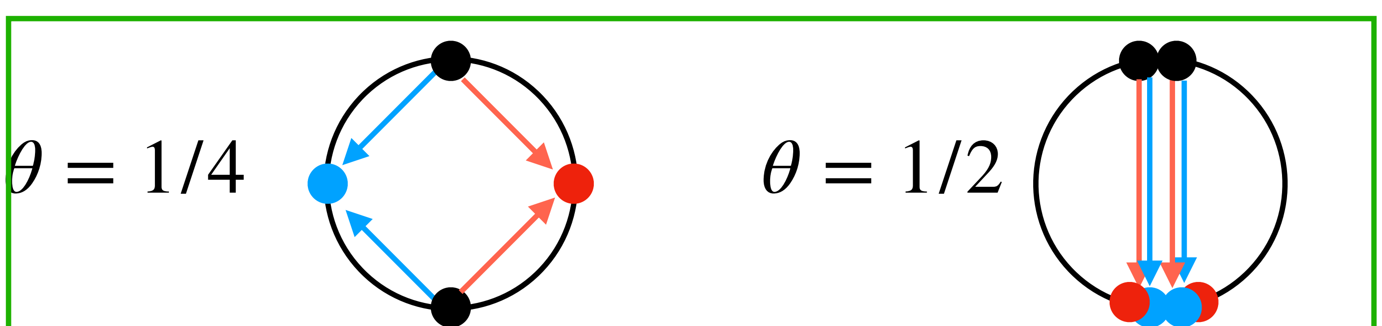
Technical Thm: For all θ, if $|A| \cdot |B| = 2^{1.01n}$, then except with exp small probability over $x \in A$,
$|\mathbb{E}_Y[\exp(2\pi i \cdot \theta \cdot \langle x, Y\rangle)]| < \exp(-\Omega(n \sin^2(4\pi\theta)))$.

$$|\mathbb{E}_Y[\exp(2\pi i \cdot \theta \cdot \langle x, Y\rangle)]|^2$$
$$= |\mathbb{E}_{Y_1}[\exp(2\pi i\theta x_1 Y_1) \cdot \mathbb{E}_{Y_{>1}}[\exp(2\pi i \cdot \theta \cdot \langle x_{>1}, Y_{>1}\rangle)]]|^2$$
$$= |\mathbb{E}_{Y_1}[\exp(2\pi i\theta x_1 Y_1) \cdot Z_{Y_1}]|^2.$$

If Y_1 has entropy, for at least **half** the choices of x_1,
$\leq \exp(-\sin^2(4\pi\theta)) \cdot \mathbb{E}_{Y_1}[|Z_{Y_1}|^2]$

If Y_1 has no entropy,
$\leq \mathbb{E}_{Y_1}[|Z_{Y_1}|^2]$





Technical Thm: For all θ, if $|A| \cdot |B| = 2^{1.01n}$, then except with exp small probability over $x \in A$,
$$|\mathbb{E}_Y[\exp(2\pi i \cdot \theta \cdot \langle x, Y \rangle)]| < \exp(-\Omega(n \sin^2(4\pi\theta)))\,.$$

$$|\mathbb{E}_Y[\exp(2\pi i \cdot \theta \cdot \langle x, Y \rangle)]|^2$$
$$= |\mathbb{E}_{Y_1}[\exp(2\pi i \theta x_1 Y_1) \cdot \mathbb{E}_{Y_{>1}}[\exp(2\pi i \cdot \theta \cdot \langle x_{>1}, Y_{>1} \rangle)]]|^2$$
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If Y_1 has no entropy,
$$\leq \mathbb{E}_{Y_1}[|Z_{Y_1}|^2]$$

By counting arguments:

- $\Omega(n)$ coordinates of Y have entropy
- Most x will make the **right** choice in $\Omega(n)$ of these coordinates.

Open Questions

[Erdös-Moser, Sárkozy-Szemerédi]

If the coordinates of x are distinct, $Y \in \{\pm 1\}^n$ uniform,

$$\Pr[\langle x, Y \rangle = k] \leq O(n^{-3/2})\,.$$

Thm [Our work]: If $X \in_R A \subseteq \{\pm 1\} \times \{\pm 2\} \times \ldots \{\pm n\}$, $Y \in_R B \subseteq \{\pm 1\}^n$, and $|A| \cdot |B| > 2^{1.01n}$, then

$$\Pr[\langle X, Y \rangle = k] \leq O(\sqrt{\log n} \cdot n^{-3/2})\,.$$

Conjecture: If $X \in_R A \subseteq \{\pm 1\} \times \{\pm 2\} \times \ldots \{\pm n\}$, $Y \in_R B \subseteq \{\pm 1\}^n$, and $|A| \cdot |B| > 2^{1.01n}$, then

$$\Pr[\langle X, Y \rangle = k] \leq O(n^{-3/2})\,.$$

Thanks!