The Vector Balancing Constant for Zonotopes

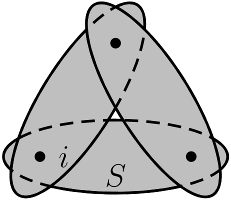
Thomas Rothvoss

Joint work with Rainie Heck and Victor Reis

Pre-Seminar Talk

UNIVERSITY *of* WASHINGTON

Discrepancy theory



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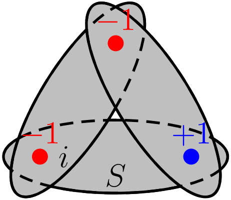

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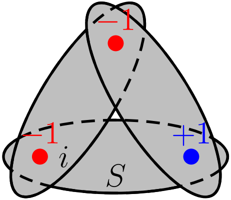

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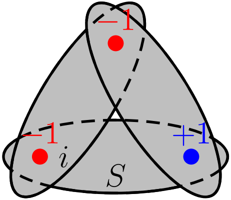

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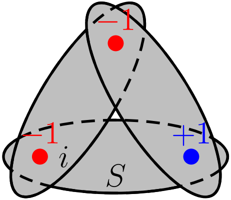

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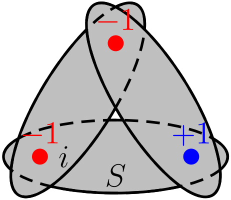

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Main method: Find a **partial coloring** $x \in \{-1, 0, 1\}^n$

- low discrepancy $\|Ax\|_\infty$
- $|\text{supp}(x)| \geq \Omega(n)$

Gaussian measure

- **Gaussian measure:**
  $\gamma_n(K) = \Pr[\text{gaussian} \in K] \approx \frac{\mathrm{Vol}_n(K \cap \sqrt{n} B_2^n)}{\mathrm{Vol}_n(\sqrt{n} B_2^n)}$

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Lemma (Sidak-Kathri '67)

For convex symmetric set K and strip S,

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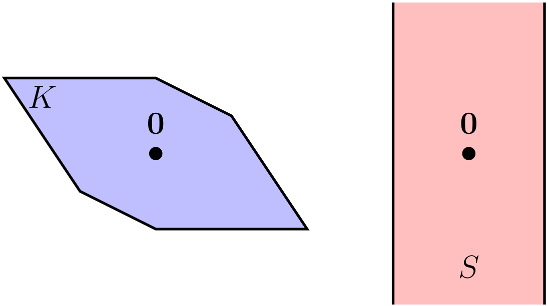

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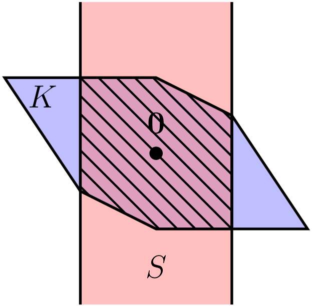

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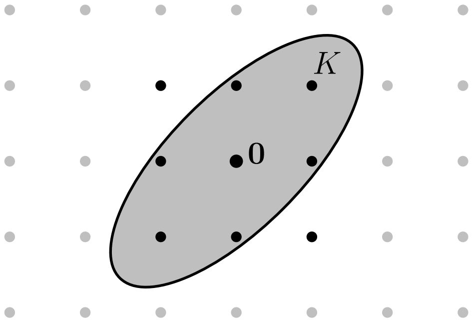
Theorem

Let K be symmetric convex set with $\gamma_n(K) \geq e^{-n/20}$. Then $\exists x \in K \cap \{-1, 0, 1\}^n$ with $|\text{supp}(x)| \geq n/10$.



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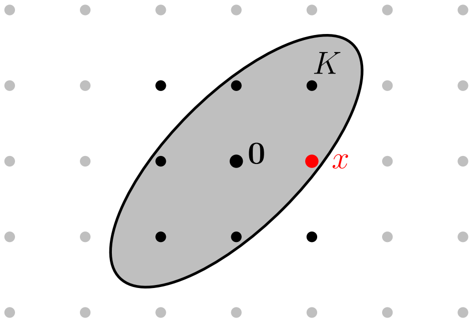

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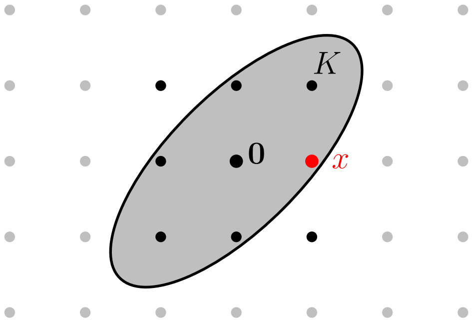
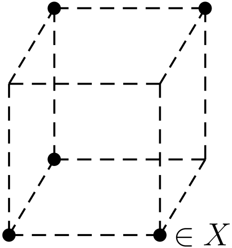


- We call an $x \in \{-1,0,1\}^n$ with $|\text{supp}(x)| \geq \frac{n}{10}$ a **good partial coloring**.

A basic fact on measure concentration

Lemma

Any set $X \subseteq \{-1,1\}^n$ *with* $|X| \geq 2^{0.8n}$ *contains* $x, y \in X$ *differing in at least* $\frac{n}{10}$ *coordinates.*



- See [Kleitman 1966] for exact bounds.

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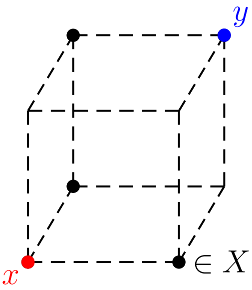

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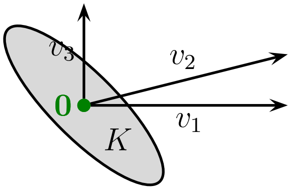
For convex symmetric set $K \subseteq \mathbb{R}^n$ *with* $\gamma_n(K) \geq e^{-n/20}$ *and* $v_1, \ldots, v_n \in \mathbb{R}^n$ *with* $\|v_i\|_2 \leq 1$, *there is a good partial coloring* x *with* $\|\sum_{i=1}^n x_i v_i\|_K \leq C$.



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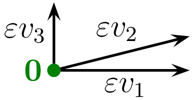

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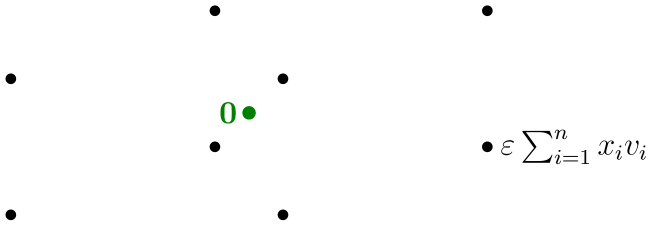

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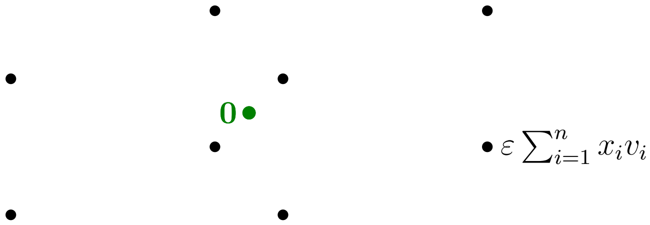

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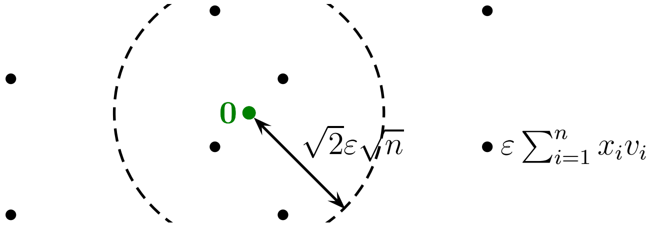

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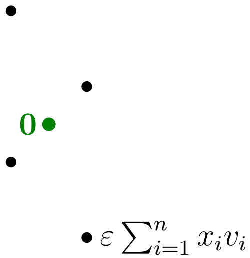

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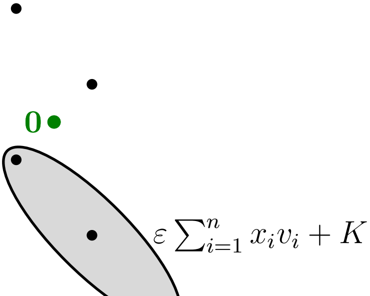

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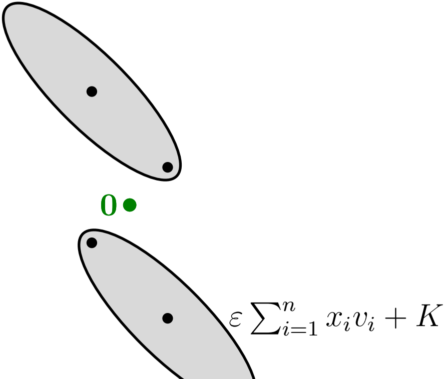

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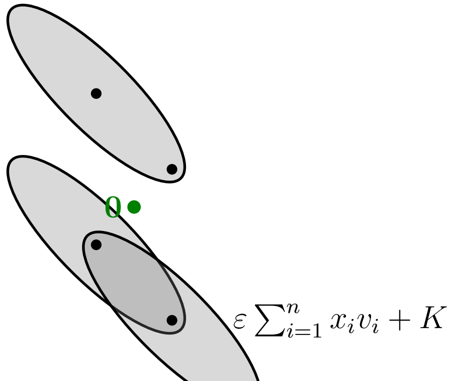

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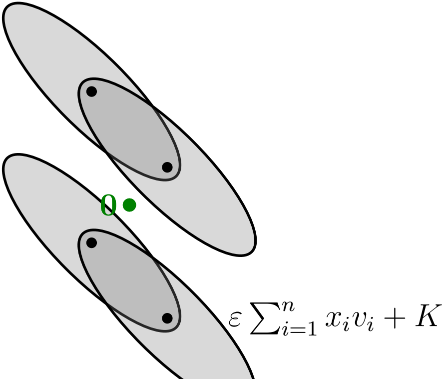

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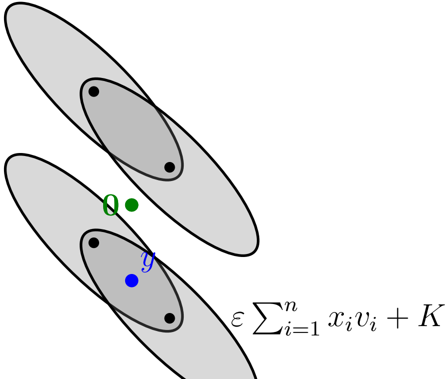

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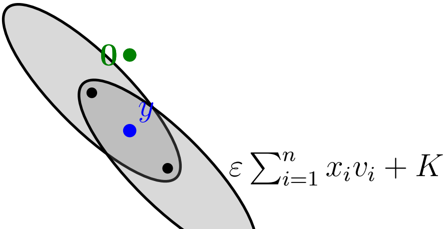

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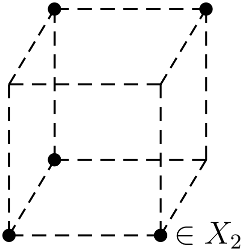

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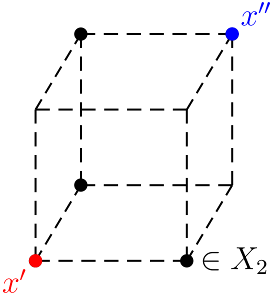

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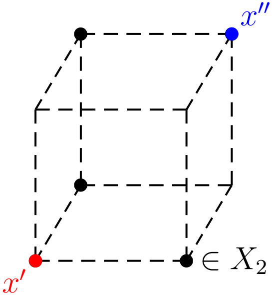

Partial colorings for Spencer

Lemma

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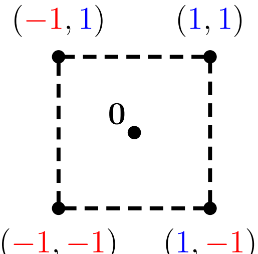

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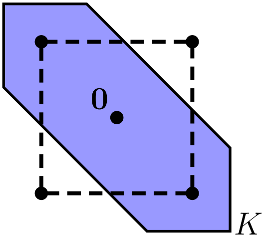

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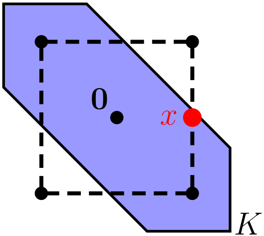

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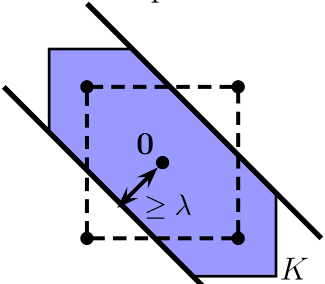
- K is intersection of m strips of width at least λ



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- K is intersection of m strips of width at least λ
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Partial colorings for Spencer

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- Set $x^* := x^1 + \ldots + x^{O(\log(n))}$. Then

$$\|Ax^*\|_\infty \leq \sum_{t \geq 0} f(n \cdot 0.9^t, m) \leq \text{const} \cdot f(n,m) \qquad \square$$

The end

Thanks for your attention

(and see you back in a few minutes..)

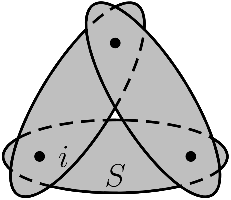
The Vector Balancing Constant for Zonotopes

Thomas Rothvoss

Joint work with Rainie Heck and Victor Reis

UNIVERSITY *of* WASHINGTON

Discrepancy theory



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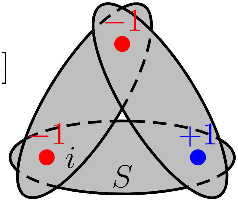

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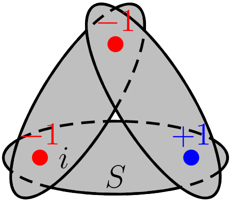

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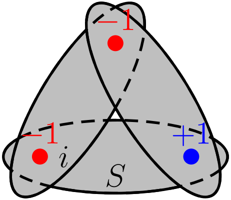

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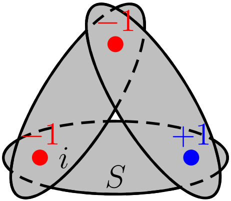

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The vector balancing constant

- For symmetric convex bodies $K, Q \subseteq \mathbb{R}^d$,

$$\mathrm{vb}(K, Q) := \sup \left\{ \min_{x \in \{-1,1\}^n} \Big\| \sum_{i=1}^{n} x_i v_i \Big\|_Q \mid n \in \mathbb{N}, v_1, \ldots, v_n \in K \right\}$$

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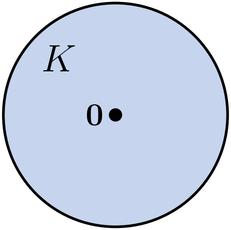

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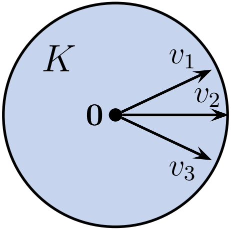

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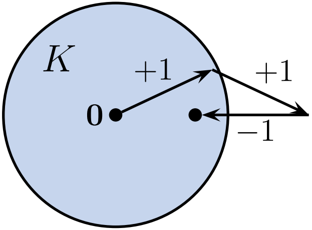

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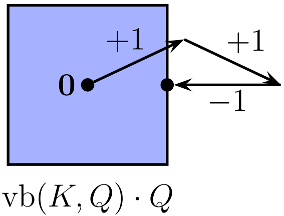

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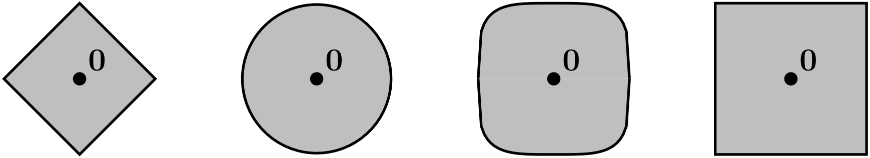
Theorem (LSV'86)

One has $vb(K, Q) \leq 2 \cdot vb_d(K, Q)$.

The L_p-balls

- Let $B_p^d := \{x \in \mathbb{R}^d \mid \|x\|_p \leq 1\}$



B_1^d B_2^d B_4^d B_∞^d

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Same bodies:

- **Spencer's Theorem.** $\mathrm{vb}(B_\infty^d, B_\infty^d) \lesssim \sqrt{d}$ and $\mathrm{vb}_n(B_\infty^d, B_\infty^d) \lesssim \sqrt{n \log \frac{2d}{n}})$ for $n \leq d$.

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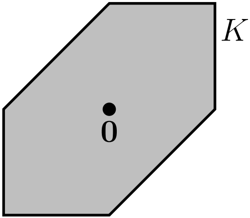
Different bodies:

- $\mathrm{vb}(B_1^d, B_\infty^d) \leq 2$ [Beck, Fiala '81]
- **Komlós Conjecture:** $\mathrm{vb}(B_2^d, B_\infty^d) \leq O(1)$
  Best known $\mathrm{vb}(B_2^d, B_\infty^d) \leq O(\sqrt{\log d})$ [Banaszczyk '98]

Zonotopes

Definition

A **zonotope** $K \subseteq \mathbb{R}^d$ is the projection of a cube.



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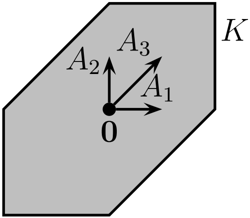

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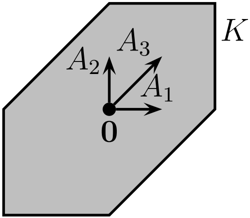

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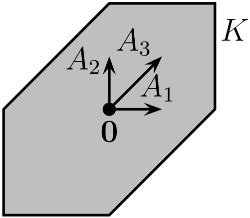

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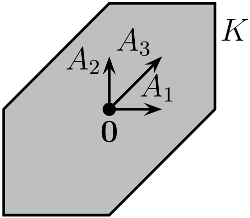

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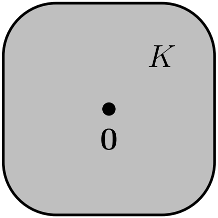
Reducing number of segments

Theorem (Talagrand '90)

For any zonotope $K \subseteq \mathbb{R}^d$ *there is a zonotope* Q *with* $O(\frac{d}{\varepsilon^2}\log(d))$ *segments so that* $Q \subseteq K \subseteq (1+\varepsilon)Q$.



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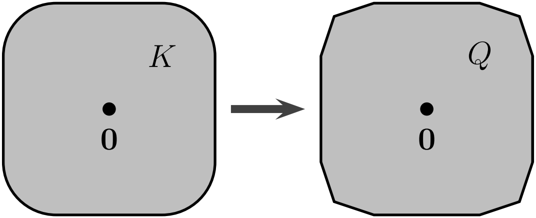

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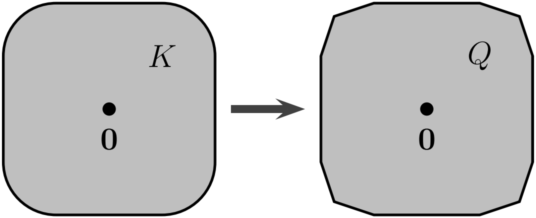

Question (Talagrand '90, Bourgain, Lindenstrauss, Milman '89))

Are $O_\varepsilon(d)$ segments enough?

vb of zonotopes – a warmup

Lemma (Folklore)

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- Then $\mathrm{vb}_d(K,K) \leq \mathrm{vb}_d(B_\infty^m, B_\infty^m) \leq O(\sqrt{d \log \frac{2m}{d}})$. $\square$

Our main contribution

Question (Schechtman; AIM workshop 2007)

Is it true that for each zonotope $K \subseteq \mathbb{R}^d$ one has $\mathrm{vb}(K,K) \leq O(\sqrt{d})$?

- Known is $\mathrm{vb}(K,K) \leq O(\sqrt{d \log\log(d)})$.

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Theorem (Heck, Reis, R. 2022)

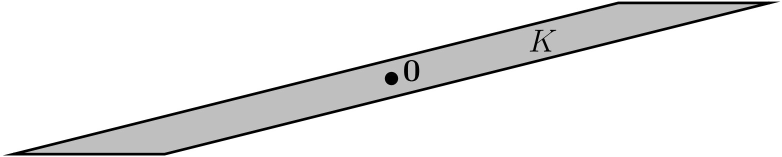
For any zonotope $K \subseteq \mathbb{R}^d$ one has $O(\sqrt{d} \log\log\log d)$.

Normalizing a zonotope

Definition

We call a zonotope $K \subseteq \mathbb{R}^d$ **normalized** if $K = \sqrt{\frac{d}{m}} A^T B_\infty^m$ where $A \in \mathbb{R}^{m \times d}$ has

- Orthonormal columns
- Short rows: $\|A_i\|_2 \leq 2\sqrt{\frac{d}{m}}$ for all i



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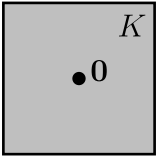

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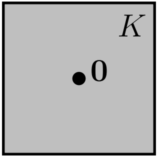
- Each zonotope can be made apx. normalized by a linear transformation + subdivision of segments (similar to [BLM' 89, Talagrand 90])
- B_∞^d is normalized



Radius of normalized zonotope

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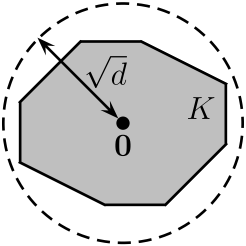
A normalized zonotope K has radius at most $\sqrt{d}$.



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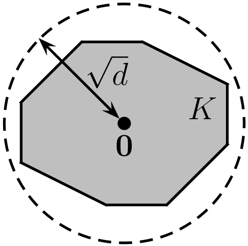

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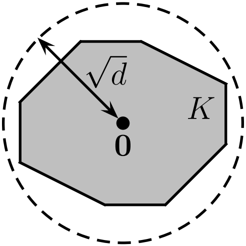

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- Then

$$\Big\|\sqrt{\frac{d}{m}}A^T y\Big\|_2 \leq \sqrt{\frac{d}{m}} \cdot \underbrace{\|A^T\|_{\text{op}}}_{\leq 1} \cdot \underbrace{\|y\|_2}_{\leq \sqrt{m}} \leq \sqrt{d}$$

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- We say $x \in [-1, 1]^n$ is a **good partial coloring** if $|\{j \in [n] : x_j \in \{-1, 1\}\}| \geq \frac{n}{2}$.

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Any symmetric convex body $P \subseteq \mathbb{R}^n$ with $\gamma_n(P) \geq e^{-C_1 n}$ and let $v_1, \ldots, v_n \in B_2^n$. Then there is a good partial coloring x with $\sum_{i=1}^n x_i v_i \in C_2 \cdot P$.

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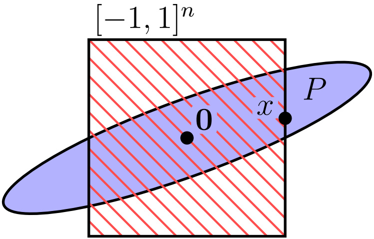
Any symmetric convex body $P \subseteq \mathbb{R}^n$ with $\gamma_n(P) \geq e^{-C_1 n}$ and let $v_1, \ldots, v_n \in B_2^n$. Then there is a good partial coloring x with $\sum_{i=1}^n x_i v_i \in C_2 \cdot P$.

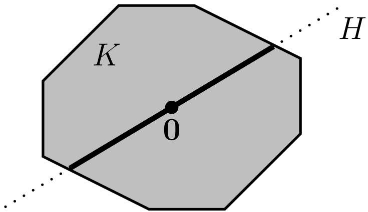
- Picture for case $v_i = e_i$:



Main technical contribution

Theorem (Heck, Reis, R. 2022)

For any normalized zonotope $K \subseteq \mathbb{R}^d$ and any n-dimensional subspace $H \subseteq \mathbb{R}^d$ one has $\gamma_H(K \cap H) \geq e^{-\Theta(n)}$.



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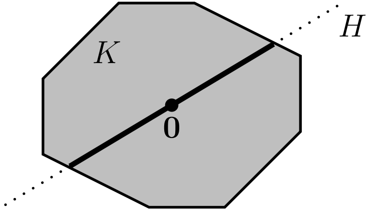

Corollary

For any $v_1, \ldots, v_n \in K$ there is a good partial coloring x so that $\sum_{i=1}^n x_i v_i \in O(\sqrt{d}) \cdot K$.

- **Proof.** Use $\|v_i\|_2 \leq \sqrt{d}$. Then use partial col. lemma with $H := \text{span}\{v_1, \ldots, v_n\}$. □

A first weak bound

Lemma

Let $K = A^T B_\infty^m$ *where columns of* A *orthonormal and let* $H \subseteq \mathbb{R}^d$ *with* $n = \dim(H)$*. Then* $\gamma_H(K \cap H) \geq e^{-n}$.

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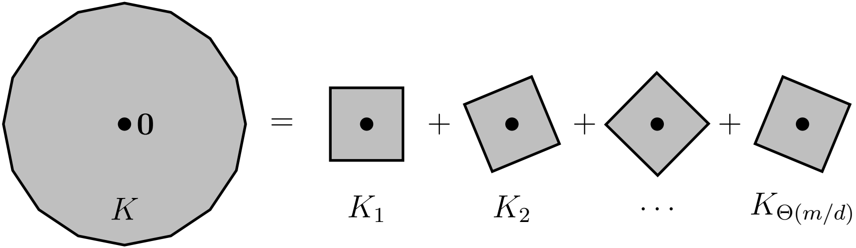
&= \exp\Big(-\underbrace{\textstyle\sum_{i=1}^m \|\Pi_H(A_i)\|_2^2}_{=n}\Big) = e^{-n}
\end{aligned}$$

Decomposing a zonotope

Lemma

Each normalized zonotope K can be written as Minkowski sum of $\Theta(\frac{m}{d})$ zonotopes K_j s.t. $\Theta(\frac{m}{d}) \cdot K_j$ is approx. normalized.*



* of the form $\tilde{A}^T B_\infty^{\tilde{m}}$ with $\sum_i \tilde{A}_i \tilde{A}_i^T \succeq \Omega(1) I_d$.

Decomposing a zonotope (2)

Theorem (Kadison-Singer problem - Marcus, Spielman, Srivastava 2015)

Let $v_1, \ldots, v_m \in \mathbb{R}^d$ *so that* $\sum_{i=1}^m v_i v_i^T = I_d$ *and* $\|v_i\|_2^2 \leq \varepsilon$ *for all* $i \in [m]$*. There is a partition* $[m] = S_1 \dot{\cup} S_2$ *so that for both* $j \in \{1, 2\}$ *one has*

$$\left(\frac{1}{2} - 3\sqrt{\varepsilon}\right) I_d \preceq \sum_{i \in S_j} v_i v_i^T \preceq \left(\frac{1}{2} + 3\sqrt{\varepsilon}\right) I_d$$

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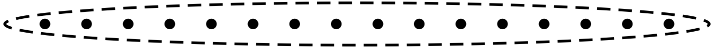

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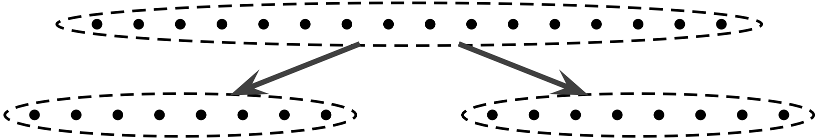

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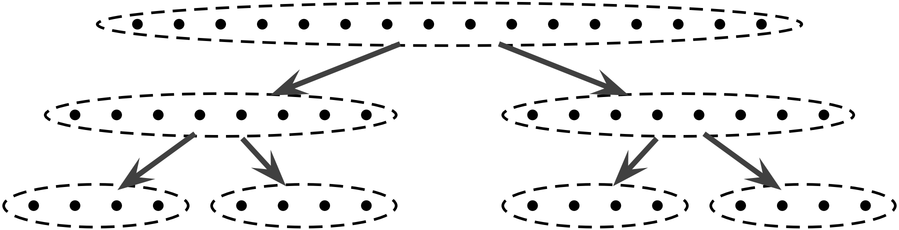

Using log concavity

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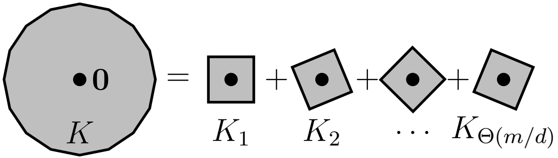

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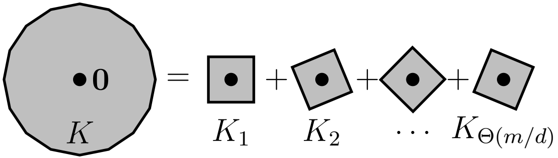

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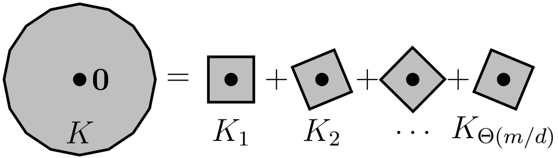

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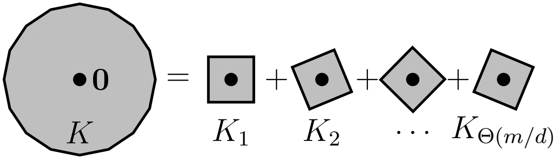

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setting $t := \log \log \log d$ and using $m \lesssim d \log d$. $\square$

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Thanks for your attention