Quantitative Lax Conjecture (w/ Raghavendra, Ryder, Weitz).

I. Context:

$$\text{LP} \underset{\neq}{\leq} \text{SDP} \overset{?}{\leq} \text{HP}$$

LP = (projections of) linear sections of $\mathbb{R}_+^m$

SDP = PSD_m

HP = hyperbolicity cones

e.g. $B_2^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\} = L \cap PSD_{n+1}$, however if $P \subset B_2^n \subset (1+\varepsilon)P$ then $m \geq \left(\frac{1}{\varepsilon}\right)^n$.

II Hyperbolic Prog

LP: $p(z_1, \dots, z_n) = z_1 z_2 \cdots z_n \in \mathbb{R}[z_1 .. z_n]$, homogeneous.

$p(\underline{1}) = \underline{1 \neq 0}$

$\forall x \in \mathbb{R}^n$, $t \longmapsto p(t\mathbb{1} - x)$ is real rooted

$\mathbb{R}^n_+ = \{ x \in \mathbb{R}^n : p(t\mathbb{1} - x)$ has all nonnegative roots$\}$.

SDP: $p(Z) = \det(Z) \qquad Z \in Sym_n$, $\binom{n}{2} + n$ vars.

$p(I) = 1 \neq 0$

$\forall X \in Sym_n$, $t \longmapsto \det(tI - X)$ is RR.

$PSD_n = \{ X \in Sym_n : \det(tI - X)$ has nonnegative roots$\}$.

Defn: $p(z_1 \ldots z_n) \in \mathbb{R}[z_1 \ldots z_n]$, homogeneous is hyperbolic w.r.t. $e \in \mathbb{R}^n$ if ① $p(e) \neq 0$

② $\forall x \in \mathbb{R}^n$, $t \mapsto p(te-x)$ is RR.

Let $K_p = \{ x \in \mathbb{R}^n : p(te-x) \text{ has all nonneg. roots} \}$

hyperbolicity cone.

Thm: [Gårding'59] K_p is a closed convex cone.

and $K_{p,e} = K_{p,e'}$ for any $e' \in K_{p,e}$.

[Guler'94]: proved self concordance of $-\log p$

HP: Optimization over sections of K_p. [Renegar]: Given oracle access to p, p', p'' there are IPM.

Key Examples : eg1 : If $A_1, \dots, A_n \succeq 0$ $\left(\sum_{i\le n} A_i \succ 0\right)$ then $p(z_1 \dots z_n) = \det\left(\sum_{i\le n} z_i A_i\right)$

is hyperbolic wrt $e = (1, 1, 1, \dots, 1)$ —.

$$K_p = \left\{ x \in \mathbb{R}^n : \sum_{i\le n} x_i A_i \succeq 0 \right\} \leftarrow \text{spectrahedral cone}$$

eg2 : Observe $D_{\underline{1}}\, z_1 \dots z_n = \sum_{i \le n} \frac{d}{dz_i}(z_1 \dots z_n) = e_{n-1}(z_1 \dots z_n)$

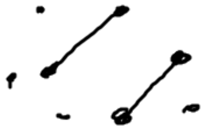
$$D_{\underline{1}}^{n-d}\, z_1 \dots z_n = \qquad = e_d(z_1 \dots z_n)$$

$\forall x \in \mathbb{R}^n$ $e_d(te - x) = \frac{d^{n-d}}{dt^{n-d}} \underbrace{e_n(te - x)}_{RR}$ RR. $K_{e_d} \supseteq K_{e_n}$

by interlacing.

<u>eg3</u>: Let $G = (V, E)$, define $\mu_G \in \mathbb{R}[x_1 \dots x_{|V|}, w_1 \dots w_{|E|}]$ by

$$\mu_G(\underline{x}, \underline{w}) = \sum_{M \subset G} \prod_{v \notin M} (-x_v) \prod_{e \in M} w_e^2$$



matchings

- Hyperbolic because of <u>Heilmann-Lieb</u> '72.
  wrt $(\underbrace{1, 1, \dots, 1}_{|V|}, 0, 0, \dots, 0)$

- $(s\underline{1}, \underline{w}) \in K_{\mu_G} \iff \lambda_{\max}(\mu_{G(w)}^{(t)}) \le s$.
  $\lambda_{\min}(\quad " \quad) \geq -s$

III Generalized Lax Conjecture [Helton]

$\sim$ [Every hyperbolicity cone is spectrahedral i.e.

$\forall p \in \mathcal{H}_{n,d}$ $\exists m$, $L \subset \mathbb{R}^{m\times m}_+$ affine subspace s.t.

hyp poly

$K_p = L \cap \mathbb{R}^{m \times m}_+$.] ☆

True for $n=3$ [Lax '58 $\rightsquigarrow$ Helton-Vinnikov '02, Lewis-Parrilo-Ramana '03]

$d=2$

Known:

- K_{e_d} is spectrahedral $(m=\binom{n}{d})$ [Branden]
- K_{M_q} " $(m=n!)$ [Amini]

- Directional deriv of det(Z) [Saunderson, Sanyal, Kummer '20]

Stronger algebraic versions of ☆ fails [Branden]

<u>Quantitative Q</u> : Given K_P, what is the least n such that

$$K_p = \mathbb{R}_+^{m\times m} \cap L \quad ?$$

<u>Defn</u>: S is an <u>η-approximate</u> spectrahedral representation of K_p

if $\quad h\,dist(S, K_p) \leq \eta$

$$\overset{\shortparallel}{\max_{\substack{\|x\|=1 \\ x\in S}} dist(x, K_p) \vee \max_{\substack{\|y\|=1 \\ y\in K_p}} dist(y, S)}$$

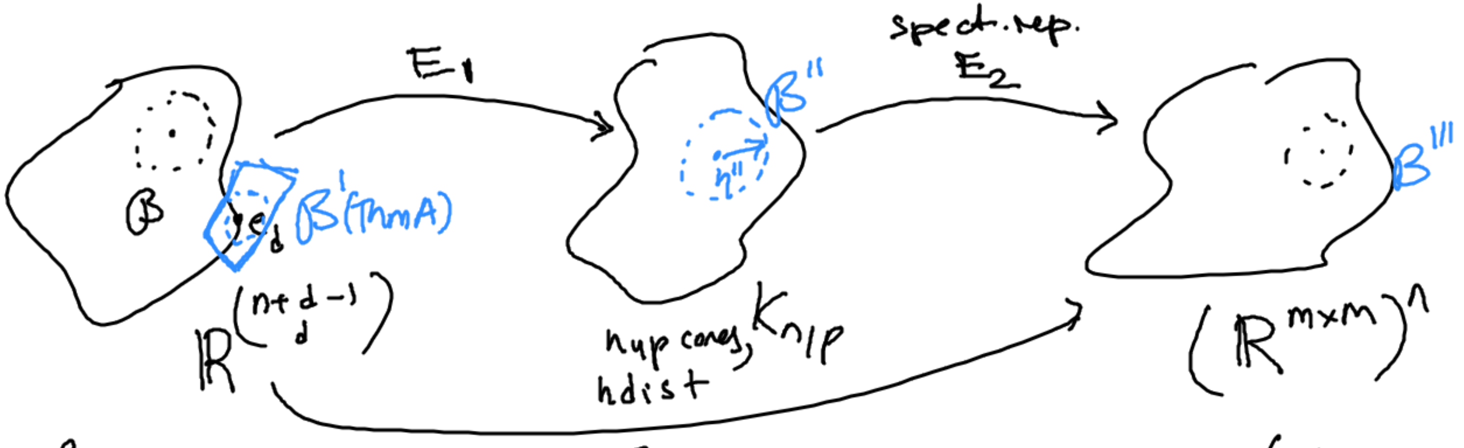
IV Results :

Thm: Let n, $d < \frac{n}{2}$, $\eta = n^{-3nd}$. Then there are (many) $p \in \mathcal{H}_{n,d}$

s.t. any η-approx spectrahedral representation of K_p must

have dimension $m \geqslant \left(\frac{n}{d}\right)^{\Omega(d)}$.

<u>Starting Point</u> [Nuij '68] $\mathcal{H}_{n,d}$ has nonempty interior in $\mathbb{R}^{\binom{n+d-1}{d}}$.



<u>Ideal Proof</u>: Suppose E maps every $p \in \mathcal{B}$ to a tuple $(A_1, \dots, A_n) \in (\mathbb{R}^{m\times m})^n$ such that $K_p = \{ x \in \mathbb{R}^n : \sum_{i\le n} x_i A_i \succeq 0 \}$

Suppose E <u>differentiable</u> & <u>locally invertible</u> in $\mathcal{B}$.

Then $E(\mathcal{B})$ contains a ball.

$\therefore \dim((\mathbb{R}^{m\times m})^n) \geq \binom{n+d-1}{d} \Longrightarrow nm^2 \geq \left(\frac{n}{d}\right)^{\Omega(d)}$

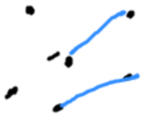
<u>Issues</u>

(1) Don't know anything about E (including existence)

(2) Need regularity to handle η-approx

<u>Theorem A</u>: Fix $n, d < n/2$. Let

(Quantitative Nuij.)



$$\boxed{S_{n,d} := \mathrm{span}\left\{ \overbrace{\prod_{ij\in M}(z_i - z_j)}^{q_M} : \begin{array}{c} M \text{ is a} \\ d\text{-matching} \\ \text{of } K_n \end{array} \right\} \subseteq} \ \mathbb{R}[z_1 \ldots z_n]$$

<u>then</u> $(e_d + S_{n,d}) \cap \mathcal{H}_{n,d}$ contains a ball $\mathcal{B}'$ of $\dim(\mathcal{B}') \geq \left(\frac{n}{d}\right)^{\Omega(d)}$ with <u>radius</u> $\geq n^{-nd}$. $= \eta'$

$$\left[\mathcal{B}' = \left\{ e_d + \sum_{M \text{ matching}} \alpha_M q_M : \|\alpha_M\|_2 \leq n^{-nd} \right\}.\right]$$

<u>Remark</u>: A polynomial $p \in \mathrm{Int}(\mathcal{H}_{n,d}) \iff \forall x$, $p(te - x)$ has <u>distinct</u> roots.
However, $e_d(t\mathbb{1} - x)$ has a double root whenever $\boxed{x \text{ has an entry repeated } \geq n-d+2 \text{ times}}$.

<u>Upshot</u>: e_d has good squeezes.

Theorem B: If $q, q' \in \mathcal{B}'$ then

$$n^{-nd} \|q - q'\|_2 \leq \text{hdist}(K_q, K_{q'}) \leq C_{n,d}(\|q - q'\|_2)$$

Consequence: $\mathcal{B}'' = E_1(\mathcal{B}')$ contains $\geq \left(\frac{n}{d}\right)^{\Omega(d)}$ cones

with pairwise hdist $\geq n^{-3nd}$.

Normalization Lemma: If there is a map $E_2 : \mathcal{B}'' \longrightarrow (\mathbb{R}^{m\times m})^n$ such that $E_2(K)$ is an η-approx. spect. rep. of K then

(**uses**: every $K \in \mathcal{B}''$ contains $\mathbb{R}^+_n$)

E_2 is bilipschitz: $\forall K, K' \in \mathcal{B}''$

$$\Omega(\| \quad , \quad \|_2) \leq \text{hdist}(K, K') \leq O(\|(A_1 .. A_n) - (A_1' .. A_n')\|_2$$

<u>Consequence</u>: $(\mathbb{R}^{m\times m})^n$ contains $\geq \left(\frac{n}{d}\right)^{\Omega(d)}$ tuples

$(A_1,\dots A_n)$ pairwise separated by $\eta = n^{-3d}$,

with $\|(A_1,\dots,A_n)\|_2 \leq 1$.

$$\|(A_1,\dots,A_n)\|_2 = \sqrt{\sum \|A_i\|_F^2}$$

$\therefore$

$$\left(\frac{1}{n^{-3nd}}\right)^{m^2 n} \geq \left(\frac{n}{d}\right)^{\Omega(d)}$$

$$\longrightarrow \quad m^2 n \log n \cdot nd \geq \left(\frac{n}{d}\right)^{\Omega(d)}$$

$$\longrightarrow \quad m \geq \left(\frac{n}{d}\right)^{\Omega(d)}.$$

V Idea of Proof of Thm A

Goal: $\left(e_d + \sum_M \alpha_M q_M\right)(t\mathbb{1} - x)$ is RR $\forall x \in \mathbb{R}^n$

Need: $\sum_M \alpha_M q_M(te - x) = 0$

whenever $\boxed{e_d(te-x)}$ has a double root.

trans. invar.

$$\underbrace{\sum_M \alpha_M q_M(x)}_{\text{perturbation.}}$$

Suppose $x_i = x_j$ whenever $i,j \in S$

$|S| \geq n-d+2$

S

Sufficient: $\left|\sum_M \alpha_M q_M(x)\right| \lesssim_{n,d} \text{mingap}\left(e_d(t\mathbb{1}-x)\right)$

Every d-matching must have an edge inside S.

$\Longrightarrow$ perturbation is zero.

Thm B: Use further combinatorics of matchings

$\Longrightarrow$ identifies a restriction ~~of~~ $X_{q,q'}$

along which max root of

$q(te - x_{q,q'})$ $\quad$ $q'(te - x_{q,q'})$

differ significantly.

Jacobi Polynomials.

VI Open Q

(Q1) Improve η.

(Q2) Projections of Sections?

[Parrilo–Saunderson: K_{e_d} have poly(n,d) SDP]

(Q3) Need $\left(\frac{n}{d}\right)^{\Omega(d)}$ bits to write down examples. poly(n,d) bits?

projections

odd degree q? [R. Oliveira '20] If VP≠VNP then M_q requires

$m \geq 2^n$ for $\eta = 0$.