Lifting for Simplicity: Concise Descriptions of Convex Sets

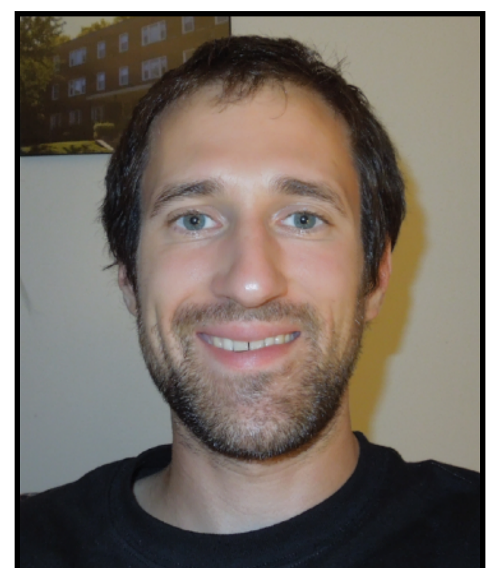
Rekha R Thomas
University of Washington

Joint work with:

Hamza Fawzi
U Cambridge

Joao Gouveia
U Coimbra

Pablo Parrilo
MIT



Richard Robinson
Nvidia

James Saunderson
Monash U

Linear Programming

nonnegative orthant: $\mathbb{R}^m_+$

linear program:

$$\begin{aligned} \max \qquad & c^\top x \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

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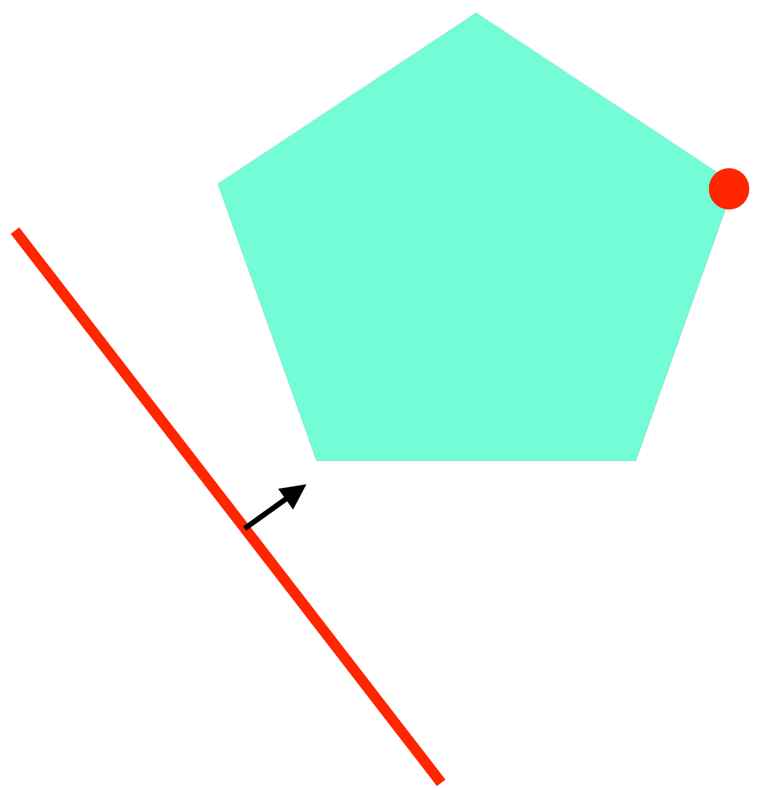

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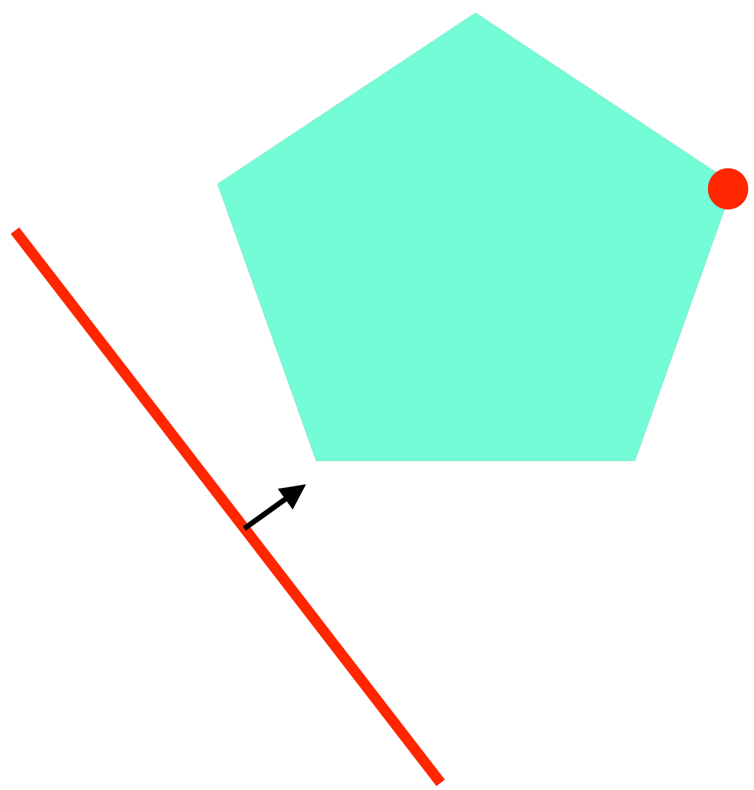
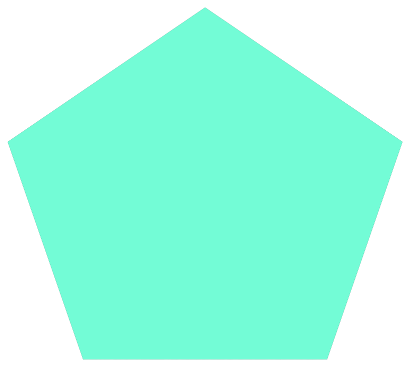
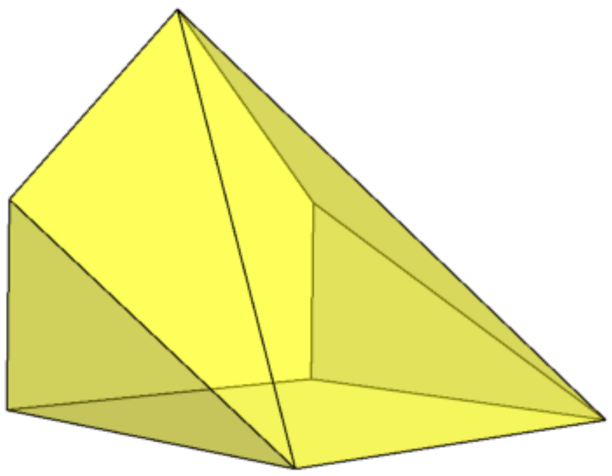
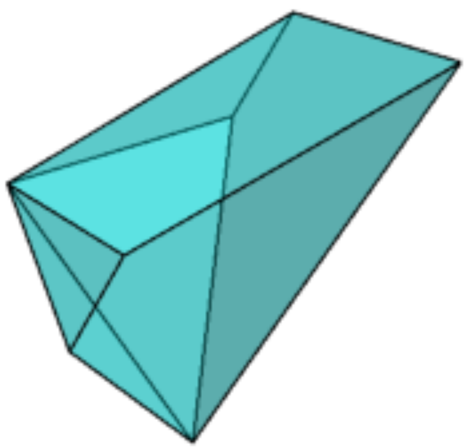


polyhedra: *feasible regions of linear programs (slices of positive orthants by affine spaces)*

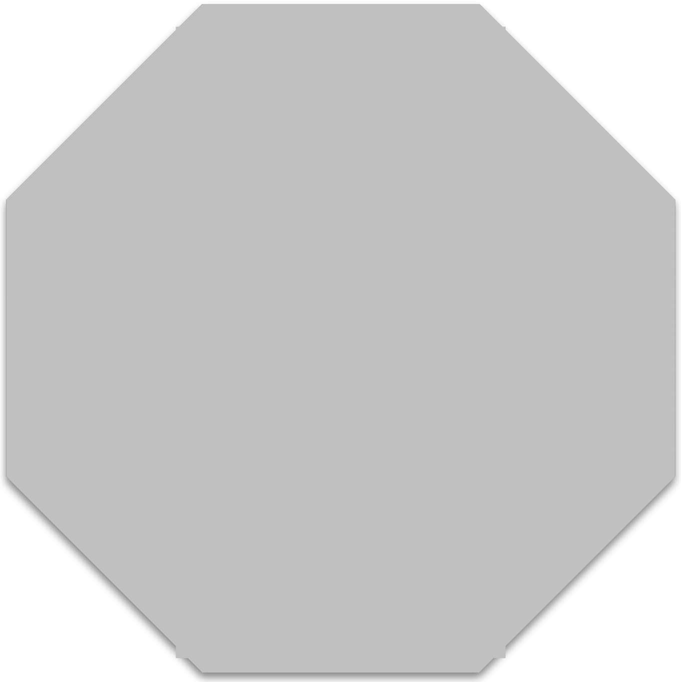






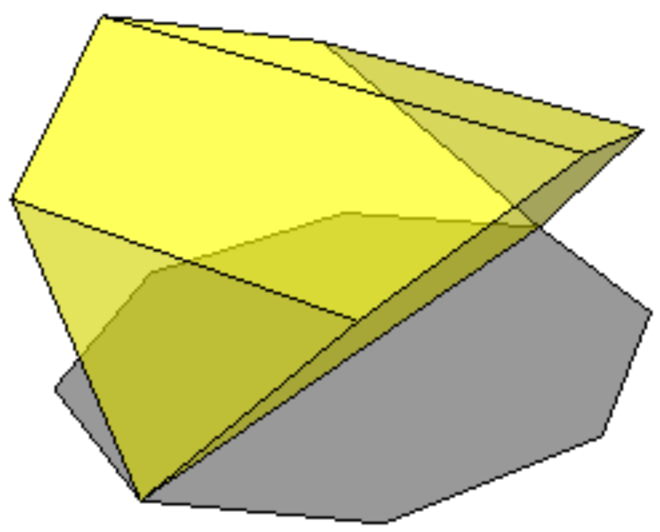
Polygons

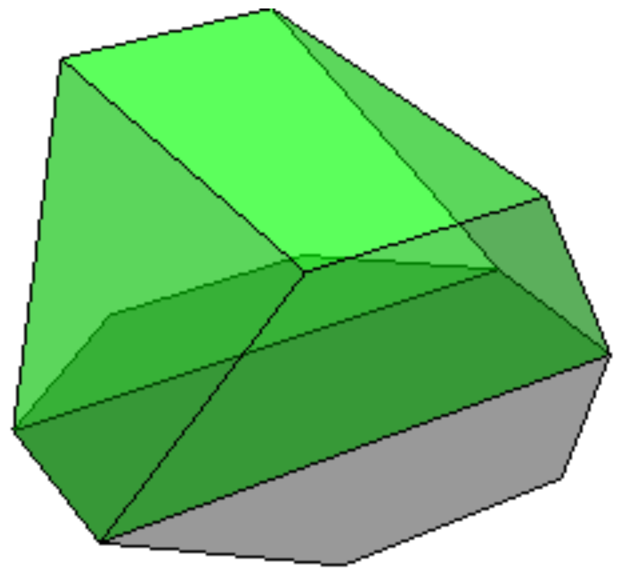
What is the most efficient linear representation of an octagon?



Polygons

An octagon is the projection of a polytope with 6 facets!





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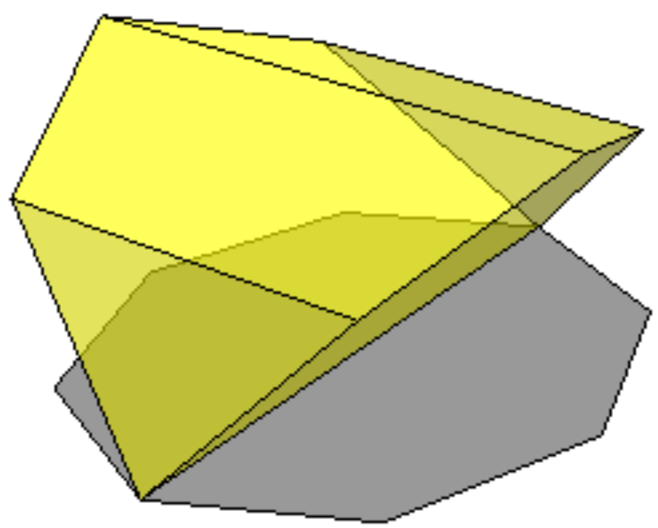
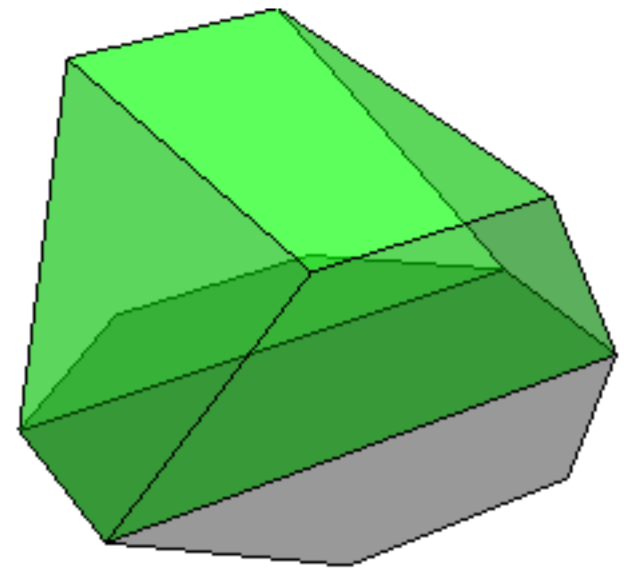


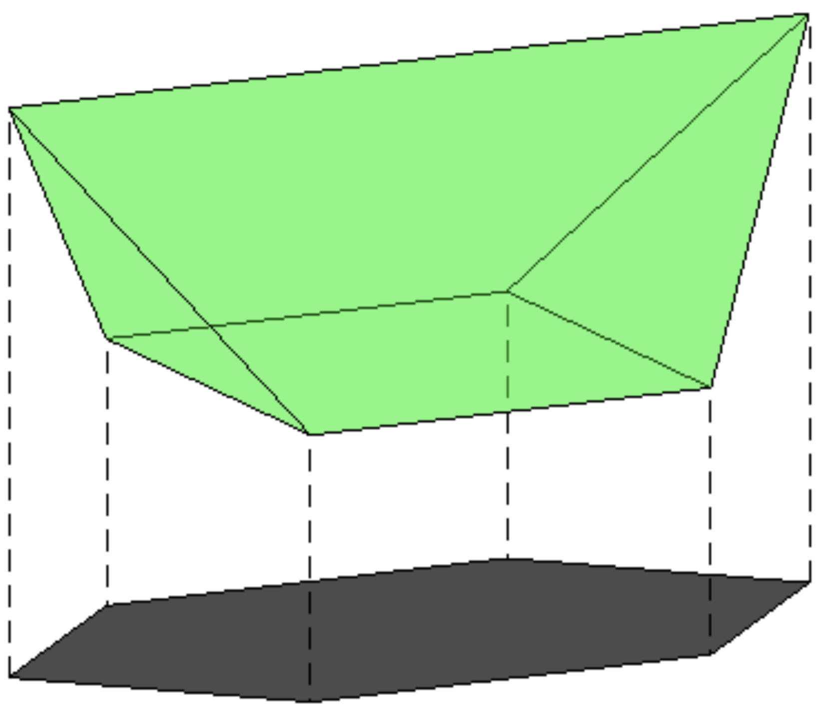


Ben-Tal Nemirovski (2001): A regular 2^n-gon is the projection of a polytope with $2n+4$ facets

Now extended to all regular n-gons, need about

$$2\lceil \log_2 n \rceil \text{ facets}$$

Polyhedral Lifts of Polytopes



lift of P

affine slice of $\mathbb{R}^m_+$

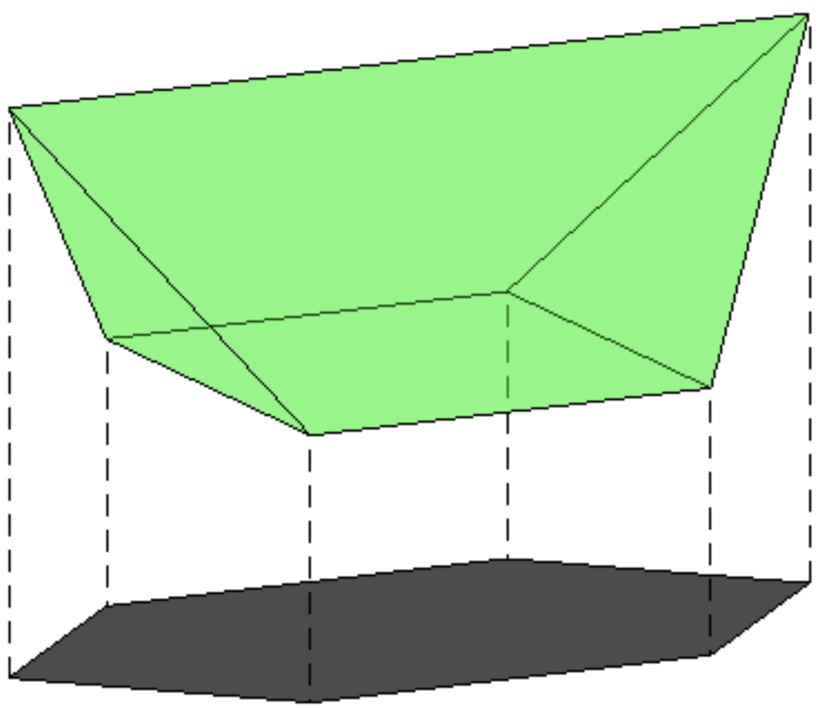
$\big\downarrow \pi$

P

polytope

$$P = \pi(\mathbb{R}^m_+ \cap L)$$

Polyhedral Lifts of Polytopes



lift of P $\quad\quad$ *affine slice of* $\mathbb{R}^m_+$

$\downarrow \pi$

P $\quad\quad$ *polytope*

$$P = \pi(\mathbb{R}^m_+ \cap L)$$

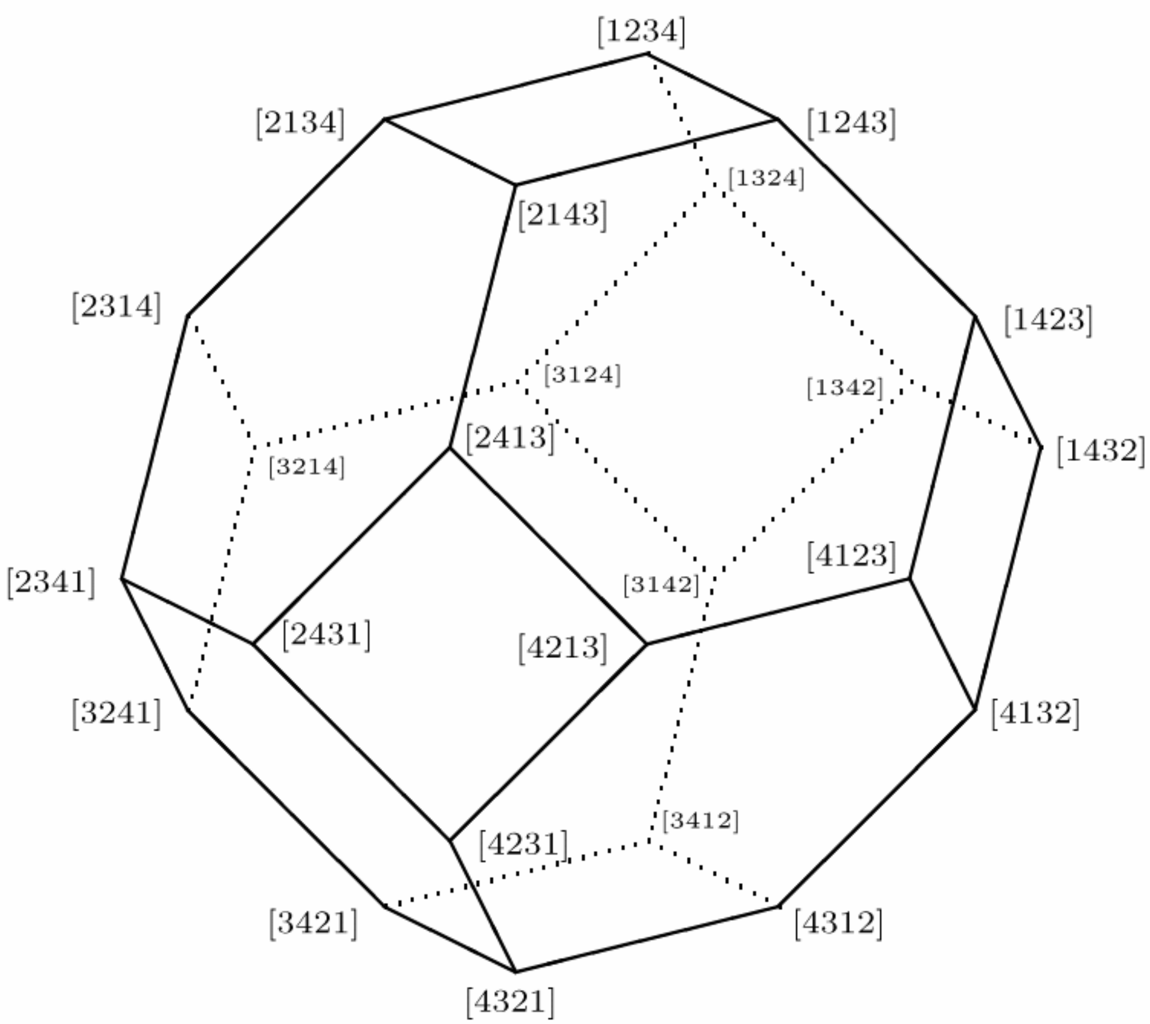
Can optimize a linear function over the lift instead of over P

Permutahedron

$\Pi_n = \mathrm{conv}(\text{permutations of } (1, 2, \ldots, n))$

$n!$ vertices

$2^n - 2$ facets



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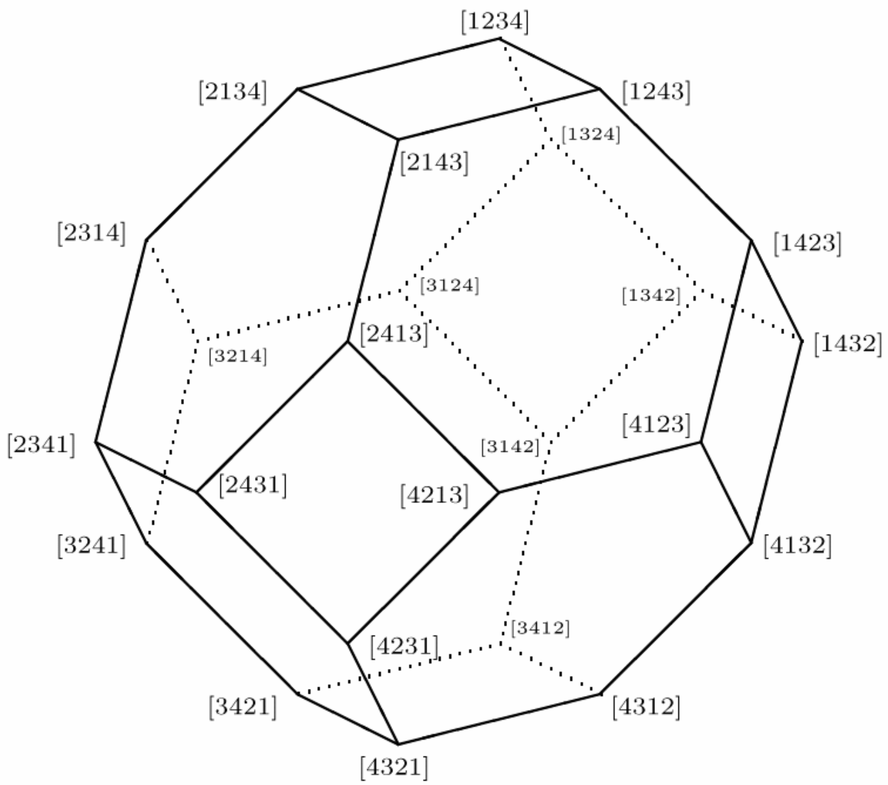

Birkhoff polytope

$$B_n = \left\{ X \in \mathbf{R}^{n\times n} \;:\; \begin{array}{l} \sum_j X_{ij} = 1,\ \forall\ i \in [n], \\ \sum_j X_{ij} = 1,\ \forall\ j \in [n], \\ X_{ij} \geq 0\ \forall\ 1 \leq i,j \leq n \end{array} \right\}$$

doubly stochastic matrices

$$\Pi_n = \pi(B_n)$$

$O(n^2)$-lift

Permutahedron

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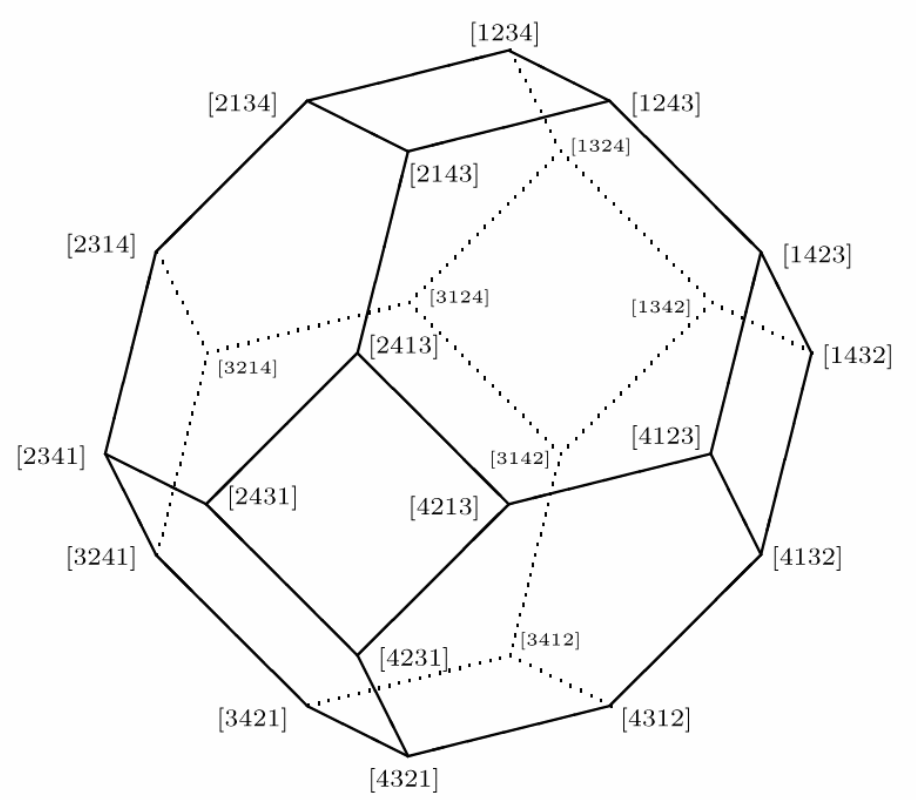

Goemans (2015):
Π_n has a $O(n \log n)$-lift

Birkhoff polytope

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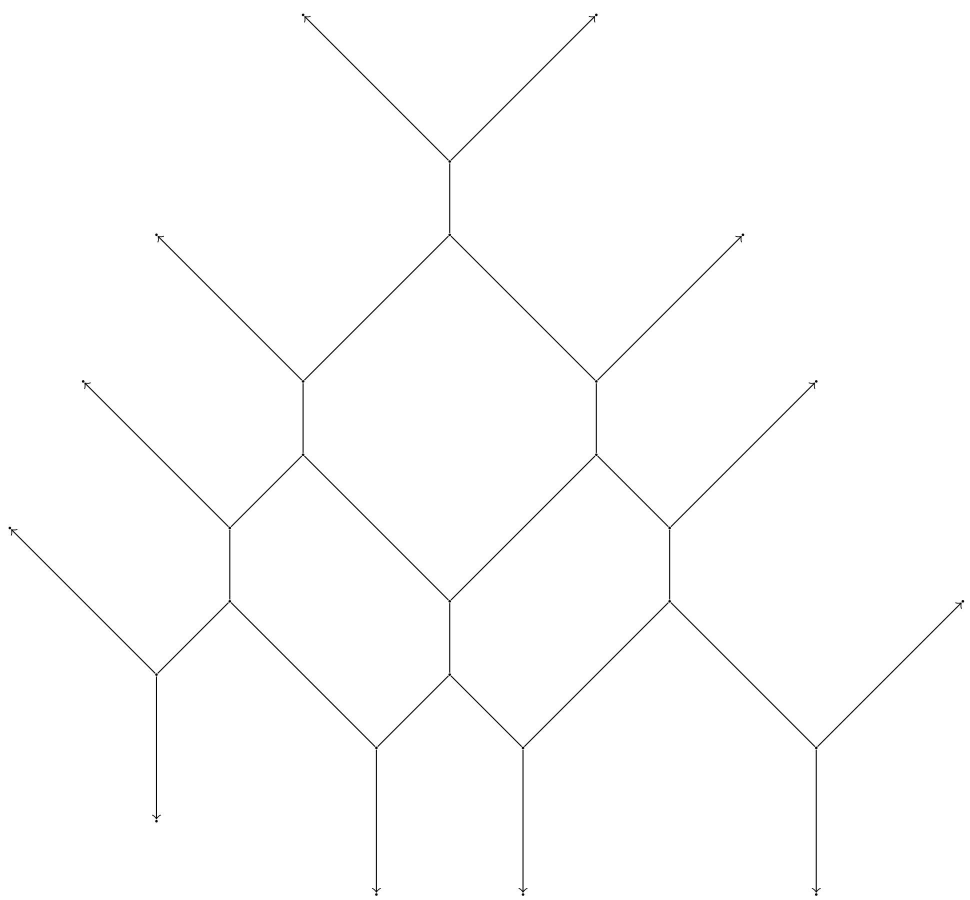
doubly stochastic matrices

$\Pi_n = \pi(B_n)$

$O(n^2)$-lift

Honeycomb lift of the Horn cone

$$\begin{aligned}\mathrm{Horn}(n) := \{(\lambda,\mu,\nu)\in(\mathbb{R}^n)^3\ :\\ \exists A,B \text{ Hermitian } n\times n \text{ with}\\ \lambda=\mathrm{eig}(A),\quad \mu=\mathrm{eig}(B),\\ \nu=\mathrm{eig}(-(A+B))\}\end{aligned}$$



Knutson-Tao (1999)

$O(n^2)$-lift

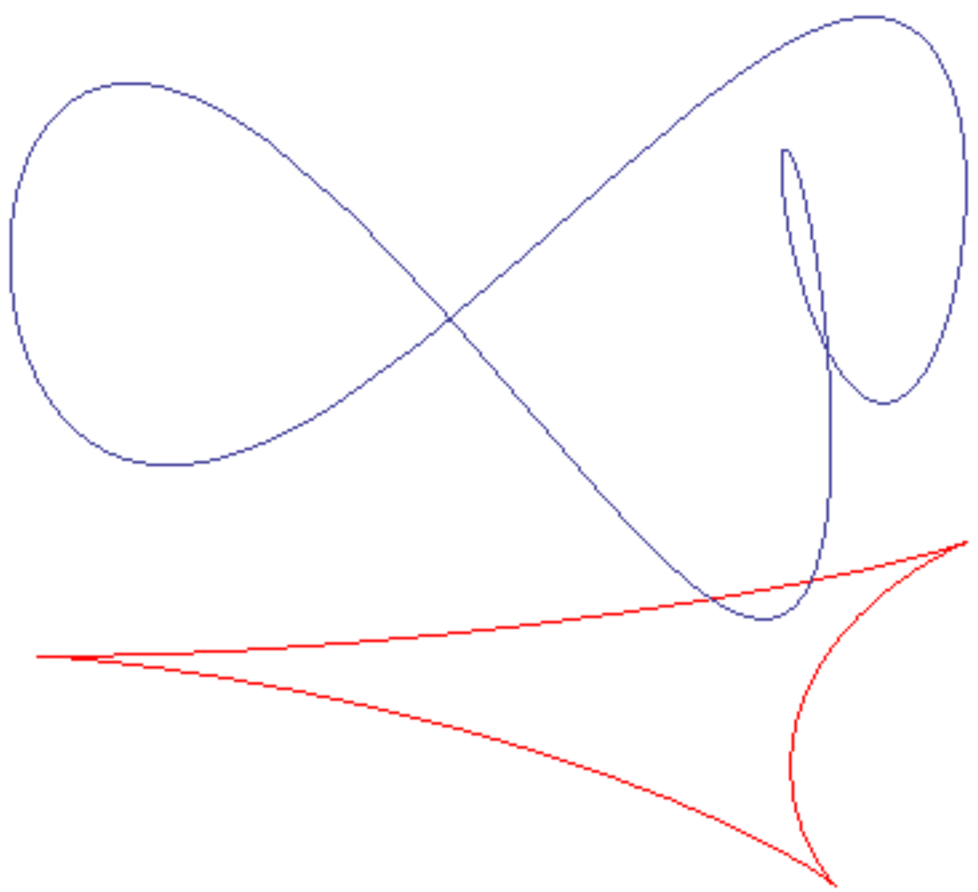
- *allows a polynomial time check whether a triple is a Horn triple*
- *connections to representation theory of GL(n)*

Resolution of Singularities

Hironaka (1964)

Y

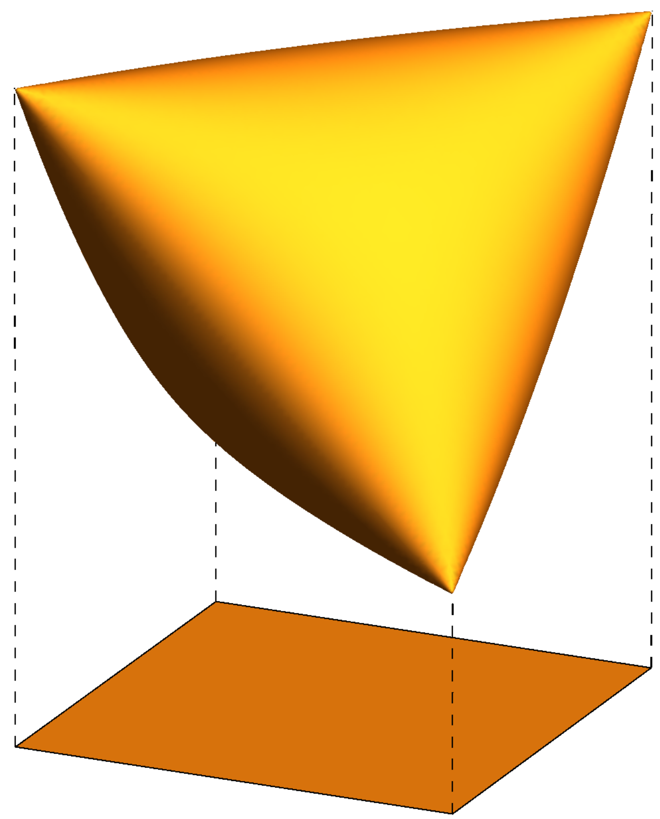
X



$$X = \{(1 + 2\cos(t) - 2\sin(t)^2, 2\sin(t) - 2\sin(t)\cos(t)) \ : \ t \in [0, 2\pi]\}$$

$$Y = \{(1 + 2\cos(t) - 2\sin(t)^2, 2\sin(t) - 2\sin(t)\cos(t), \sin(3t)) \ : \ t \in [0, 2\pi]\}$$

Conic Lifts of Convex Sets



lift of C

affine slice of a closed convex cone K

$\Big\downarrow \pi$

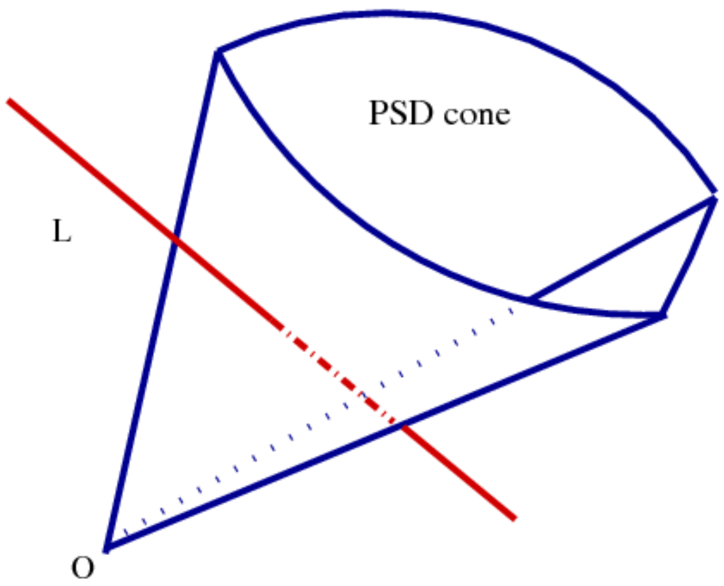
$$\boxed{C = \pi(K \cap L)}$$

C

convex set

Semidefinite Programming

psd cone: $\mathcal{S}_+^n = \{n \times n \ \ \text{psd matrices}\}$



semidefinite program:

$$\begin{aligned} \max \quad & C \cdot X \\ & A_i \cdot X = b_i \quad i = 1, \ldots, m \\ & X \succeq 0 \end{aligned}$$

Semidefinite Programming

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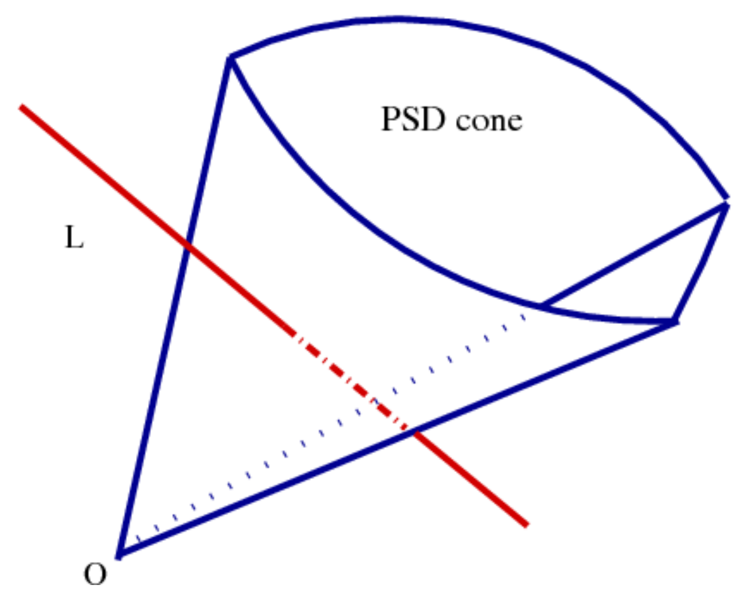
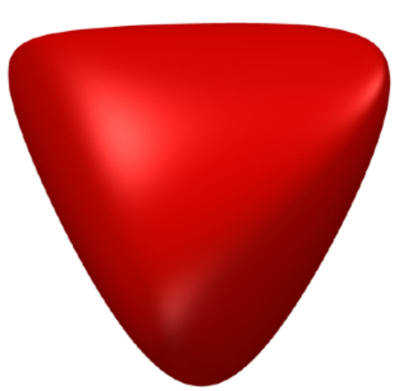
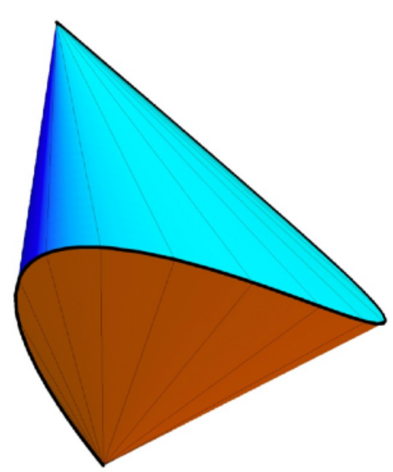
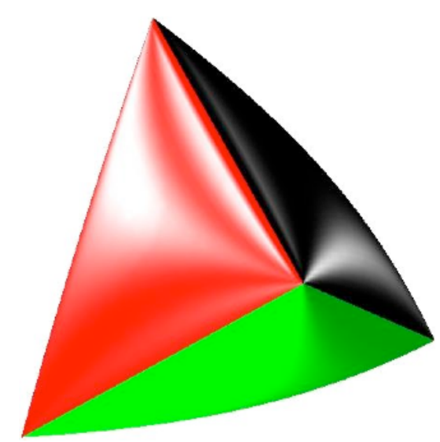
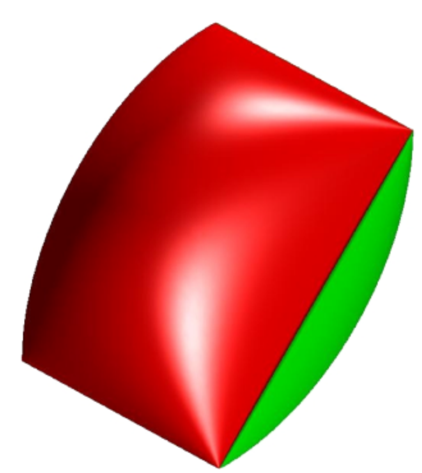


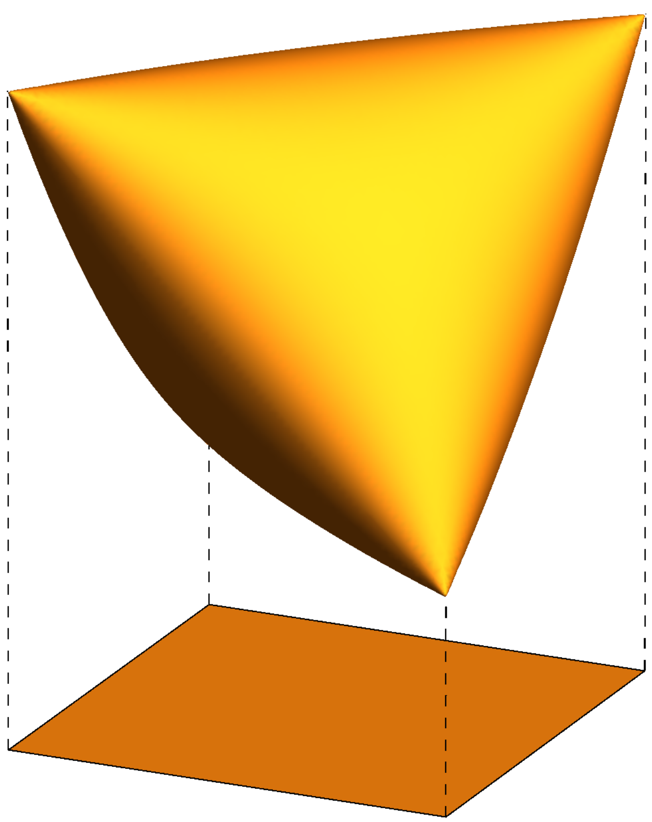
semidefinite program:

$$\begin{array}{ll} \max & C \cdot X \\ & A_i \cdot X = b_i \quad i = 1, \ldots, m \\ & X \succeq 0 \end{array}$$

spectrahedra: *feasible regions of semidefinite programs (slices of the psd cone by affine spaces)*

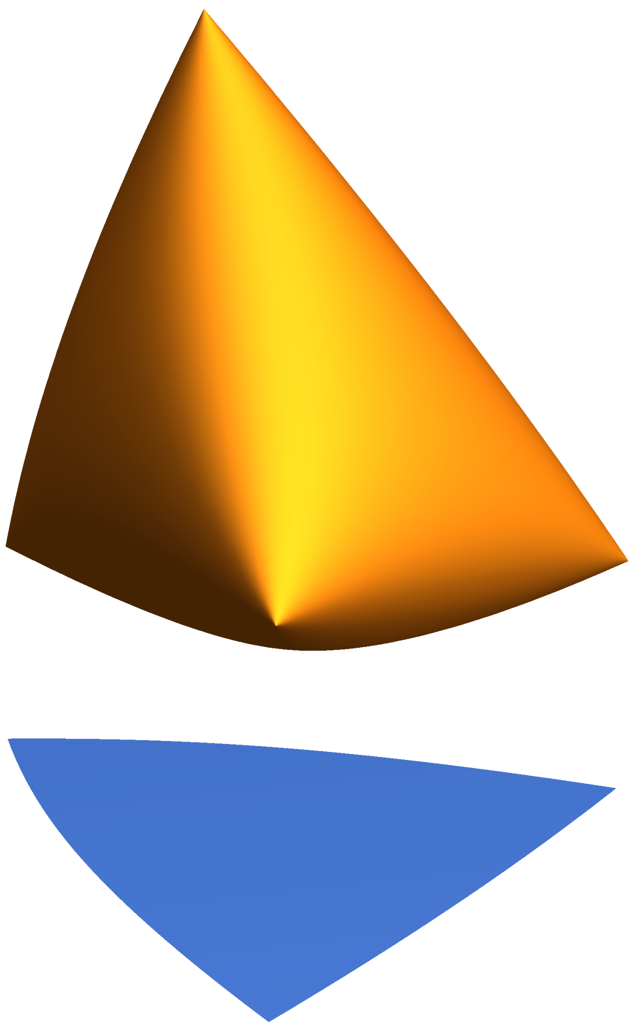
Spectrahedral Lifts of Convex Sets



K-lift of C

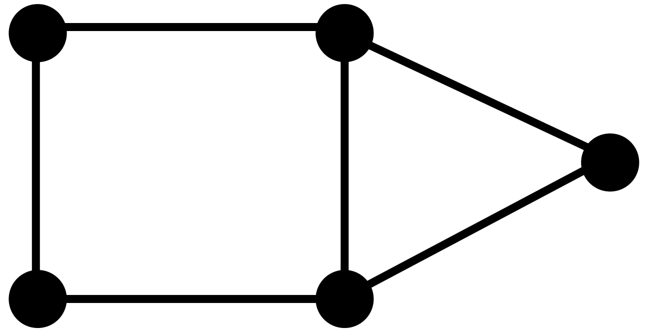
$\downarrow$

C



$$K = \mathcal{S}_+^m$$

$G = ([n], E)$



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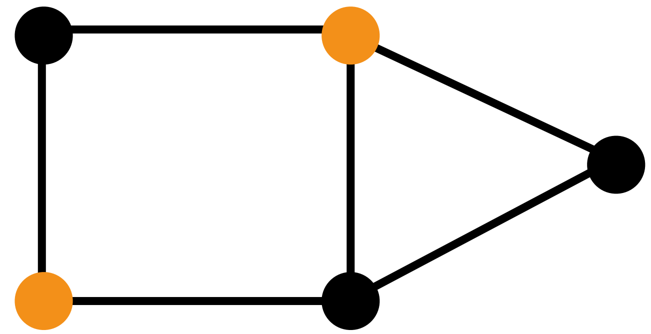

$S \subseteq [n]$ is stable if

$\forall\ i, j \in S,\ ij \notin E$

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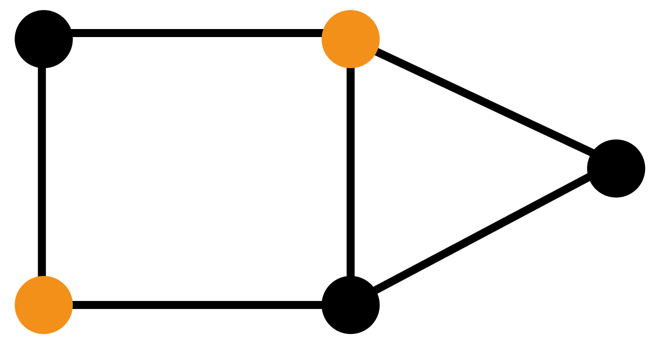

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Stable set problem:

$$\boxed{\max\ \{|S|\ :\ S \text{ stable set in } G\}}$$

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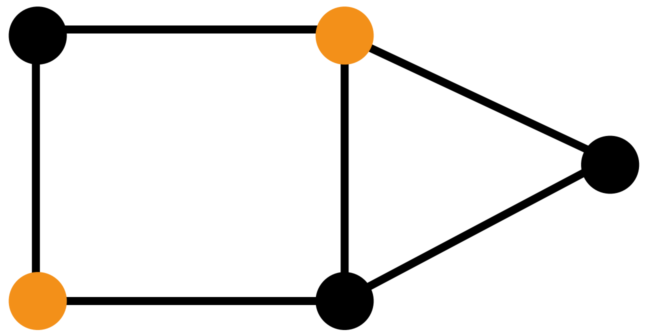

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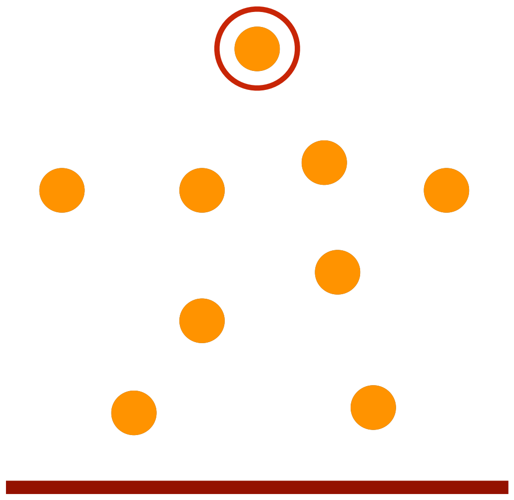
$$S \leftrightarrow \chi^S \quad \text{where} \quad (\chi^S)_i = \begin{cases} 1 \text{ if } i \in S \\ 0 \text{ if } i \notin S \end{cases}$$

$$\max\ \left\{ \sum_{i \in [n]} x_i\ :\ \begin{array}{ll} x_i^2 = x_i & \forall\, i \in [n] \\ x_i x_j = 0 & \forall\, ij \in E \end{array} \right\}$$



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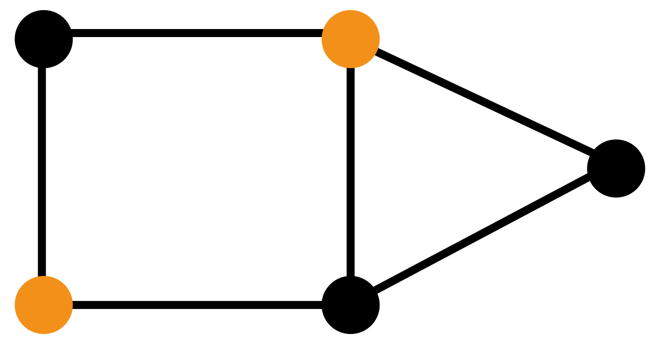

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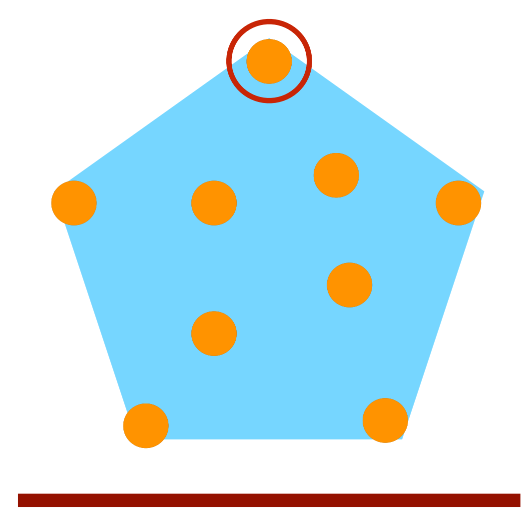
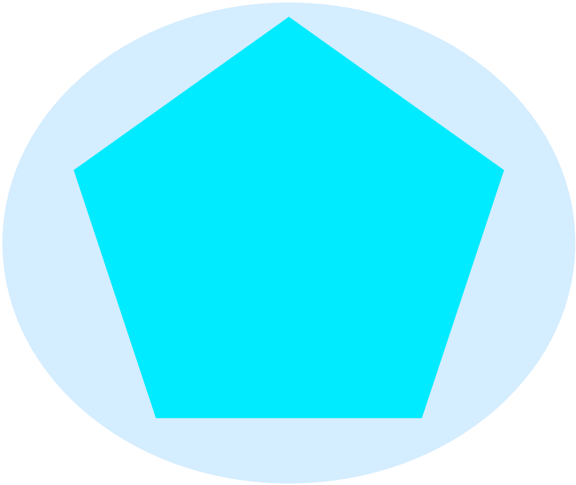


$$= \max \left\{ \sum_{i\in[n]} x_i \ :\ x \in \underbrace{\mathrm{conv}\{\chi^S\ :\ S \text{ stable }\}}_{\text{STAB(G)}} \right\}$$

Lovàsz theta body of a graph

$\mathrm{TH}(G)$



Lovàsz ('79): $$\mathrm{TH}(G) = \left\{ x \in \mathbf{R}^n \ : \ \exists U \text{ s.t. } \underset{U_{ii} = x_i \ \forall i, \ U_{ij} = 0 \ \forall ij \in E}{\begin{pmatrix} 1 & x^\top \\ x & U \end{pmatrix} \succeq 0} \right\}$$

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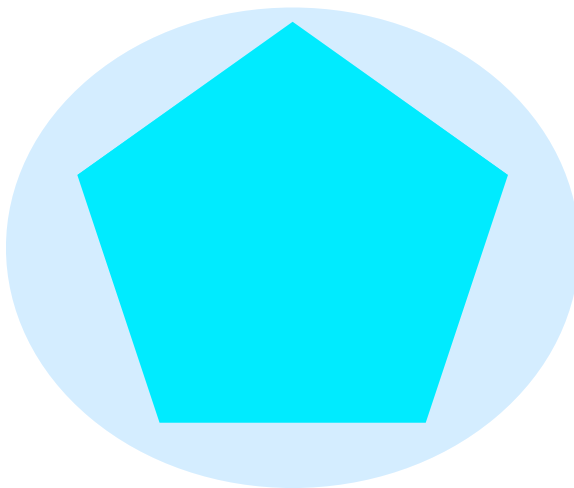

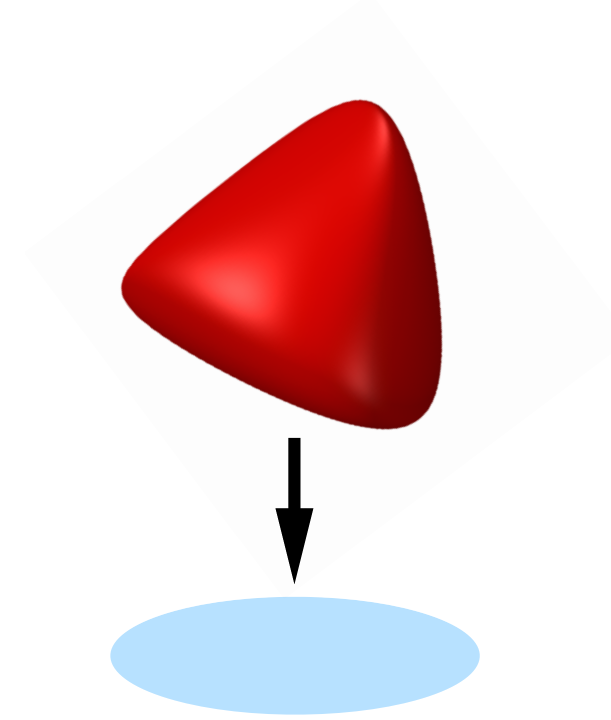

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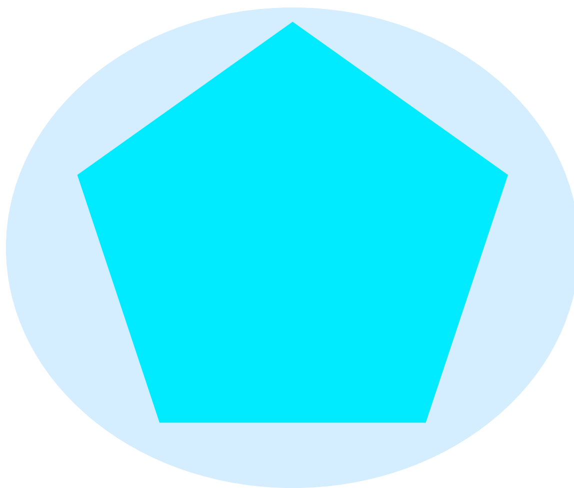
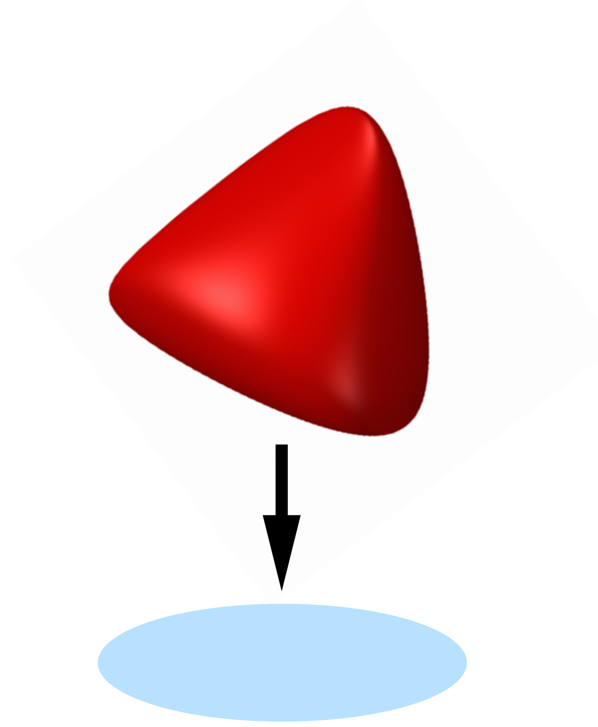
Geometry: $\mathrm{TH}(G)$ is the projection of a spectrahedron

Lovàsz theta body of a graph

$\mathrm{TH}(G)$





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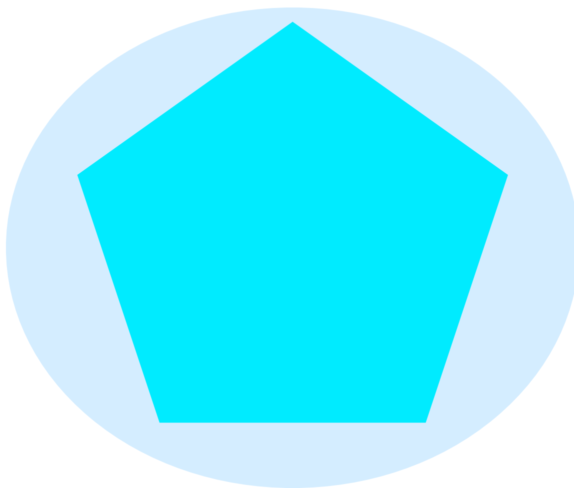
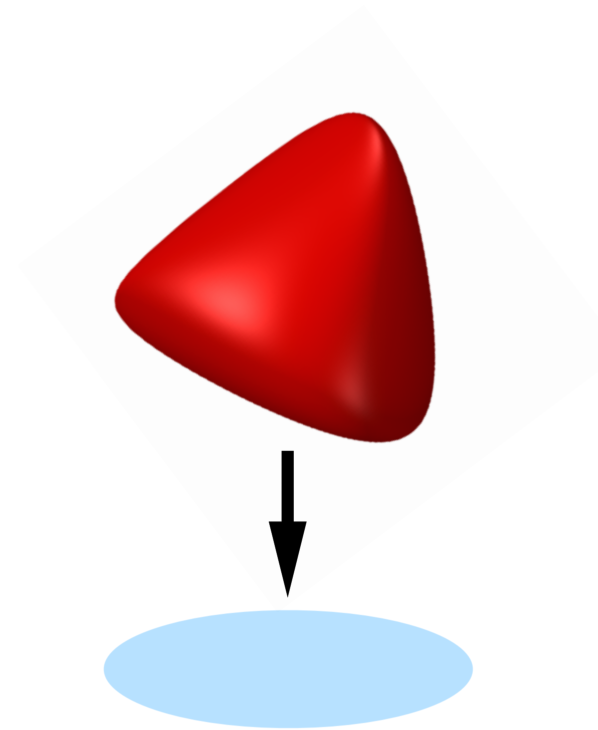
Geometry: $\mathrm{TH}(G)$ is the projection of a spectrahedron

Complexity: Can optimize over $\mathrm{TH}(G)$ in polynomial time using semidefinite programming

Lovàsz theta body of a graph

$\mathrm{TH}(G)$





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Geometry: $\mathrm{TH}(G)$ is the projection of a spectrahedron

Complexity: Can optimize over $\mathrm{TH}(G)$ in polynomial time using semidefinite programming

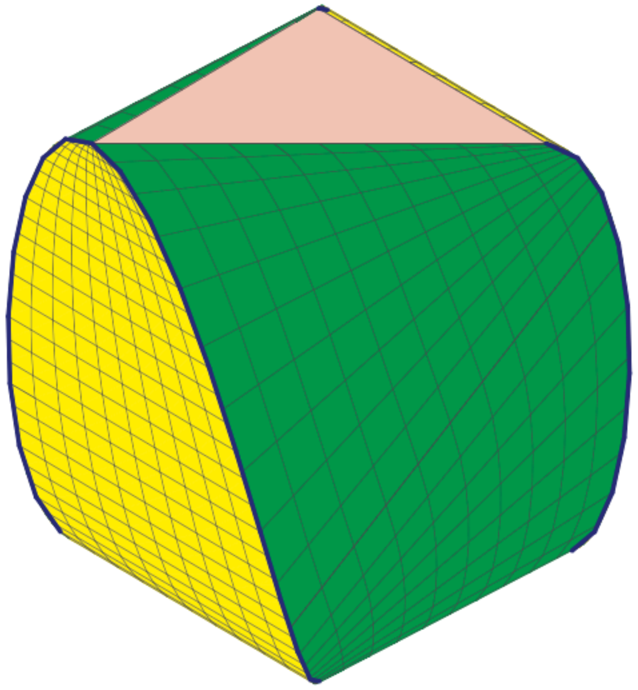
Exactness: $\mathrm{STAB}(G) = \mathrm{TH}(G) \Leftrightarrow G$ perfect graph

Ranestad-Sturmfels (2010)

$$C = \mathrm{conv}(\cos(\theta), \sin(2\theta), \cos(3\theta))$$

boundary consists of

- *2 triangles*
- *a cubic surface*
- *surface of degree 16 (green)*



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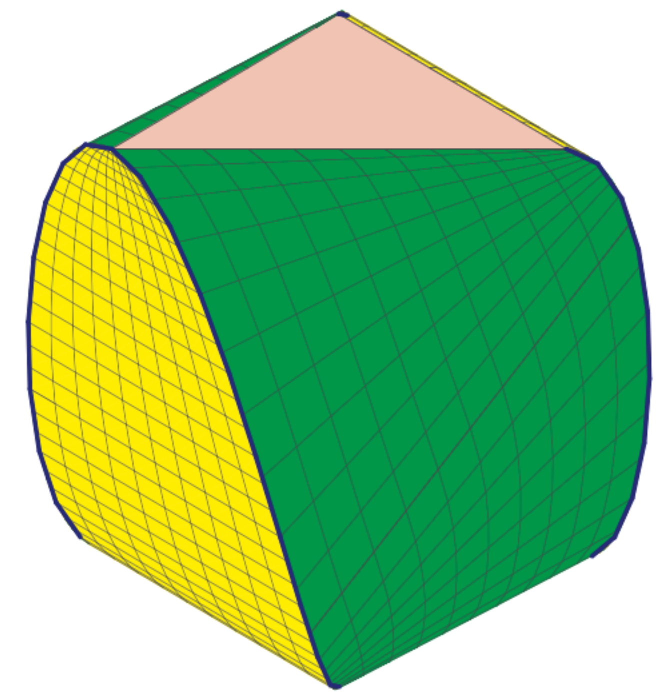

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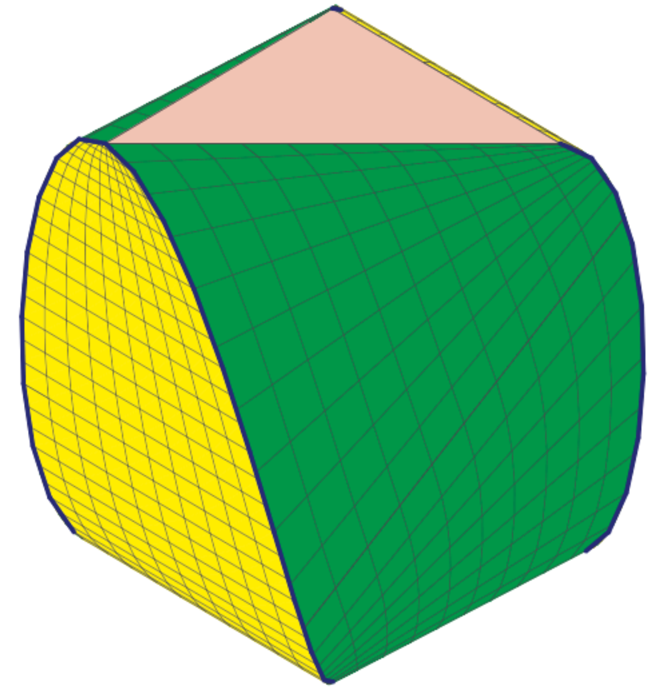
$$\begin{gathered}
1024x^{16} - 12032x^{14}y^2 + 52240x^{12}y^4 - 96960x^{10}y^6 + 56160x^8y^8 + 19008x^6y^{10} + 1296x^4y^{12} + 6144x^{15}z - 14080x^{13}y^2z \\
-72000x^{11}y^4z + 149440x^9y^6z + 79680x^7y^8z + 7488x^5y^{10}z + 15360x^{14}z^2 + 36352x^{12}y^2z^2 + 151392x^{10}y^4z^2 + 131264x^8y^6z^2 \\
+18016x^6y^8z^2 + 20480x^{13}z^3 + 73216x^{11}y^2z^3 + 105664x^9y^4z^3 + 23104x^7y^6z^3 + 15360x^{12}z^4 + 41216x^{10}y^2z^4 + 16656x^8y^4z^4 \\
+6144x^{11}z^5 + 6400x^9y^2z^5 + 1024x^{10}z^6 - 26048x^{14} - 135688x^{12}y^2 + 178752x^{10}y^4 + 124736x^8y^6 - 210368x^6y^8 + 792x^4y^{10} \\
+5184x^2y^{12} + 432y^{14} - 77888x^{13}z + 292400x^{11}y^2z + 10688x^9y^4z - 492608x^7y^6z - 67680x^5y^8z + 21456x^3y^{10}z + 2592xy^{12}z \\
-81600x^{12}z^2 - 65912x^{10}y^2z^2 - 464256x^8y^4z^2 - 192832x^6y^6z^2 + 31488x^4y^8z^2 + 6552x^2y^{10}z^2 - 40768x^{11}z^3 - 194400x^9y^2z^3 \\
-196224x^7y^4z^3 + 14912x^5y^6z^3 + 8992x^3y^8z^3 - 20800x^{10}z^4 - 84088x^8y^2z^4 - 7360x^6y^4z^4 + 7168x^4y^6z^4 - 12480x^9z^5 \\
-9680x^7y^2z^5 + 3264x^5y^4z^5 - 2624x^8z^6 + 760x^6y^2z^6 + 64x^7z^7 + 189649x^{12} + 104700x^{10}y^2 - 568266x^8y^4 + 268820x^6y^6 \\
+118497x^4y^8 - 42984x^2y^{10} - 432y^{12} + 62344x^{11}z - 592996x^9y^2z + 421980x^7y^4z + 377780x^5y^6z - 79748x^3y^8z - 18288xy^{10}z \\
+104620x^{10}z^2 + 56876x^8y^2z^2 + 480890x^6y^4z^2 - 12440x^4y^6z^2 - 51354x^2y^8z^2 - 936y^{10}z^2 + 35096x^9z^3 + 181132x^7y^2z^3 \\
+73800x^5y^4z^3 - 52792x^3y^6z^3 - 3780xy^8z^3 - 6730x^8z^4 + 52596x^6y^2z^4 - 19062x^4y^4z^4 - 5884x^2y^6z^4 + y^8z^4 + 6008x^7z^5 \\
+2516x^5y^2z^5 - 4324x^3y^4z^5 + 4xy^6z^5 + 2380x^6z^6 - 1436x^4y^2z^6 + 6x^2y^4z^6 - 152x^5z^7 + 4x^3y^2z^7 + x^4z^8 - 305250x^{10} \\
+313020x^8y^2 + 174078x^6y^4 - 291720x^4y^6 + 74880x^2y^8 + 84400x^9z + 278676x^7y^2z - 420468x^5y^4z + 20576x^3y^6z + 40704xy^8z \\
-25880x^8z^2 - 76516x^6y^2z^2 - 148254x^4y^4z^2 + 77840x^2y^6z^2 + 5248y^8z^2 - 29808x^7z^3 - 49388x^5y^2z^3 + 23080x^3y^4z^3 \\
+14560xy^6z^3 + 14420x^6z^4 - 7852x^4y^2z^4 + 9954x^2y^4z^4 + 568y^6z^4 + 848x^5z^5 + 92x^3y^2z^5 + 1164xy^4z^5 - 984x^4z^6 + 724x^2y^2z^6 \\
-2y^4z^6 + 112x^3z^7 - 4xy^2z^7 - 2x^2z^8 + 140625x^8 - 270000x^6y^2 + 172800x^4y^4 - 36864x^2y^6 - 75000x^7z + 36000x^5y^2z \\
+46080x^3y^4z - 24576xy^6z - 12500x^6z^2 + 49200x^4y^2z^2 - 19968x^2y^4z^2 - 4096y^6z^2 + 15000x^5z^3 - 10560x^3y^2z^3 \\
-3072xy^4z^3 - 2250x^4z^4 - 1872x^2y^2z^4 + 768y^4z^4 - 520x^3z^5 + 672xy^2z^5 + 204x^2z^6 - 48y^2z^6 - 24xz^7 + z^8.
\end{gathered}$$

Ranestad-Sturmfels (2010)

$$C = \operatorname{conv}(\cos(\theta), \sin(2\theta), \cos(3\theta))$$

boundary consists of

- *2 triangles*
- *a cubic surface*
- *surface of degree 16 (green)*



C is the linear image of a spectrahedron

$$\left\{ (x,y,z) \in \mathbb{R}^3 \mid \exists\, u,v,w \in \mathbb{R} : \begin{pmatrix} 1 & x+ui & v+yi & z+wi \\ x-ui & 1 & x+ui & v+yi \\ v-yi & x-vi & 1 & x+ui \\ z-wi & v-yi & x-ui & 1 \end{pmatrix} \succeq 0 \right\}.$$

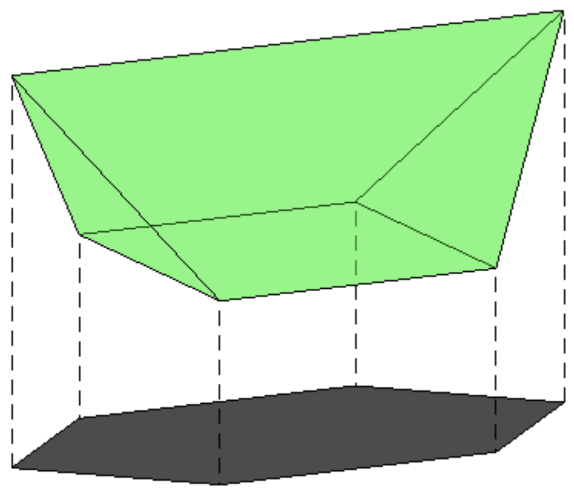
Polyhedral lifts of polytopes

Given $P \subset \mathbb{R}^n$ $K = \mathbb{R}^m_+$

polytope

when is $P = \pi(K \cap L)$?

π *linear map* L *affine space*

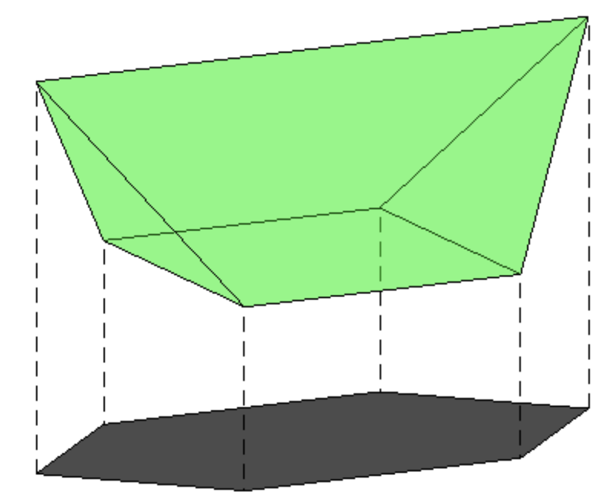


Polyhedral lifts of polytopes

Given $P \subset \mathbb{R}^n$ *polytope* $K = \mathbb{R}^m_+$

when is $P = \pi(K \cap L)$?

π *linear map* L *affine space*



$$P = \{x \in \mathbb{R}^n \; : \; Fx \leq d\}$$

v vertices f facets

$$\begin{array}{rcl} & d_j - f_j^\top x \geq 0 & \\ S_P = & \begin{pmatrix} & \vdots & \\ \cdots & d_j - f_j^\top v_i & \cdots \\ & \vdots & \end{pmatrix}_{v \times f} & v_i \end{array}$$

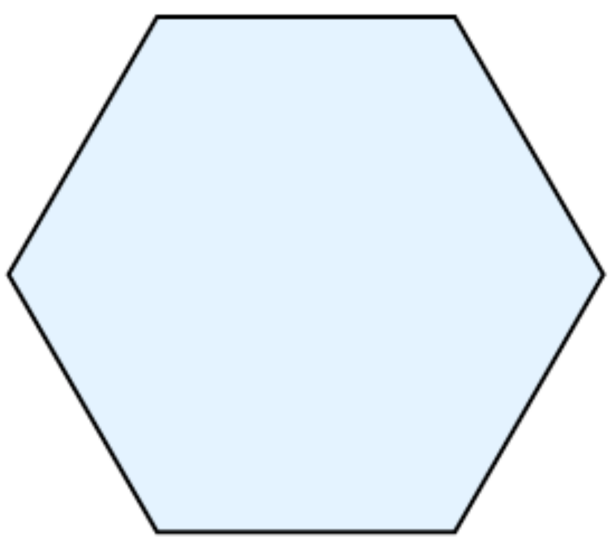
slack matrix of P

$$P = \{x \in \mathbb{R}^n \,:\, Fx \le d\}$$

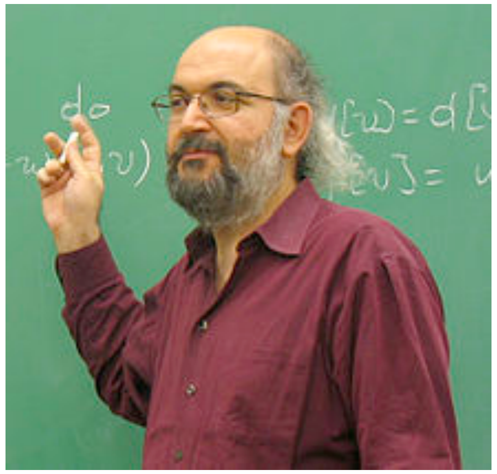
v vertices $\qquad f$ facets

$$\begin{array}{cc} d_j - f_j^\top x \ge 0 & \\ S_P = \begin{pmatrix} & \vdots & \\ \cdots & d_j - f_j^\top v_i & \cdots \\ & \vdots & \end{pmatrix}_{v \times f} & v_i \end{array}$$

slack matrix of P



$$\begin{pmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{pmatrix}$$



Yannakakis' Theorem

$$K = \mathbb{R}^m_+$$

K-factorization of S_P

$a_1, \ldots, a_v \in K$

$b_1, \ldots, b_f \in K^*$

s.t. $\langle a_i, b_j \rangle = d_j - f_j^\top v_i$

$$K^* = \{y : \langle y, x \rangle \geq 0 \ \forall \ x \in K\}$$

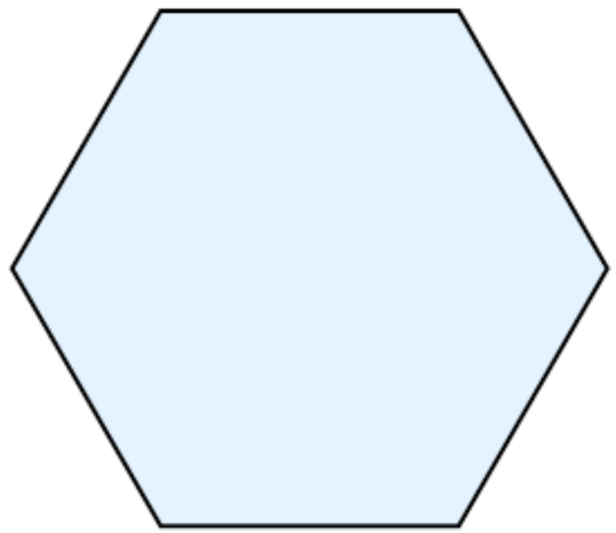
$\mathbb{R}^m_+$-factorization

nonnegative factorization

Yannakakis (1991):

P has a $\mathbb{R}^m_+$-lift $\Leftrightarrow$ S_P has a $\mathbb{R}^m_+$-factorization

Regular Hexagon

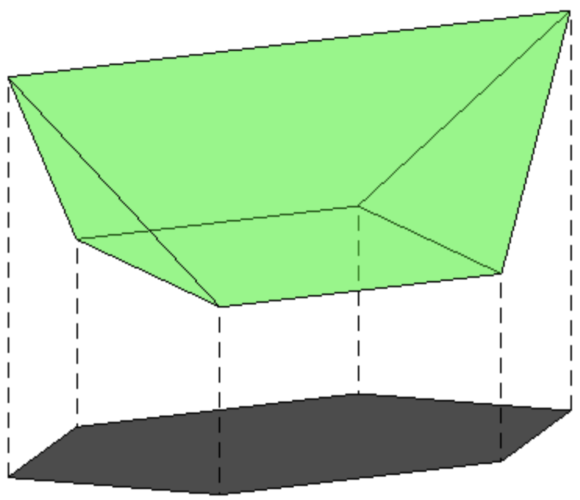


$$
\begin{array}{rcrcc}
x & + & y/\sqrt{3} & \leq & 1 \\
 & & 2y/\sqrt{3} & \leq & 1 \\
-x & + & y/\sqrt{3} & \leq & 1 \\
-x & - & y/\sqrt{3} & \leq & 1 \\
 & - & 2y/\sqrt{3} & \leq & 1 \\
x & - & y/\sqrt{3} & \leq & 1
\end{array}
$$

$$
S := \begin{pmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
$$

regular hexagon has a $\mathbb{R}^5_+$-lift

$$
\left\{ y \in \mathbb{R}^5_+ \ : \quad \begin{array}{l} y_1 + y_2 + y_3 + y_5 = 2 \\ y_3 + y_4 + y_5 = 1 \end{array} \right\}
$$



Lifts of convex sets

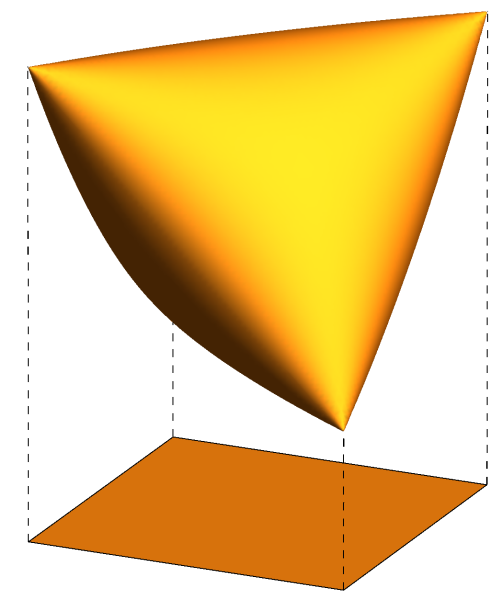
Gouveia-Parrilo-T. (2011)

Given $C \subset \mathbb{R}^n$ $K \subset \mathbb{R}^m$

convex set *convex cone*

when is $C = \pi(K \cap L)$?

π *linear map* L *affine space*

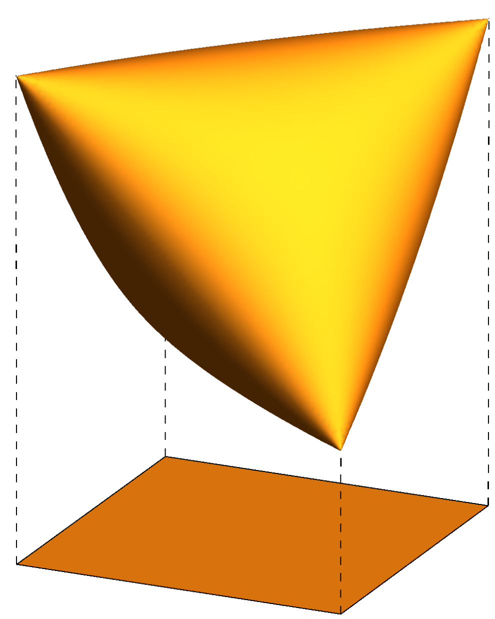


Theorem GPT (2011):

C has a K-lift $\Leftrightarrow$ S_C has a K-factorization.

generalizes Yannakakis theorem

Spectrahedral Lift of a Square



$$P = \left\{ (x, y) \in \mathbf{R}^2 \ : \ \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \succeq 0 \right\}$$

$$S_P = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$A: \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$B: \quad \frac{1}{4}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

slack operator:

$$\begin{array}{rcccc} S_C & : & \mathrm{ext}(C) \times \mathrm{ext}(C^\circ) & \to & \mathbf{R} \\ & & (x, y) & \mapsto & 1 - \langle x, y \rangle \end{array}$$

$$C^\circ = \{y \, : \, \langle x, y \rangle \leq 1 \; \forall \; x \in C\}$$

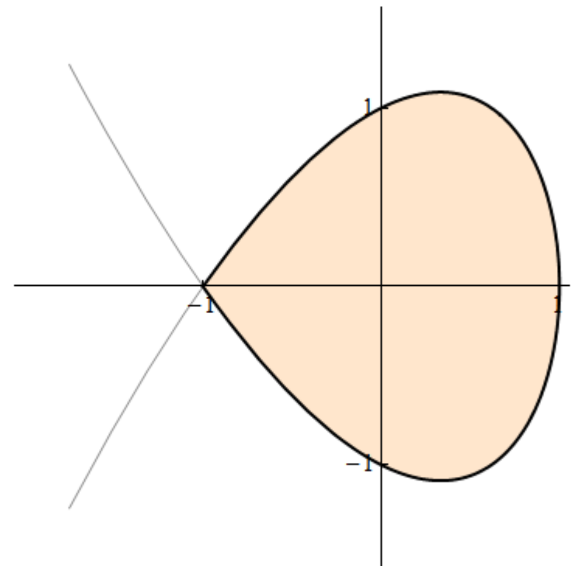
slack operator:

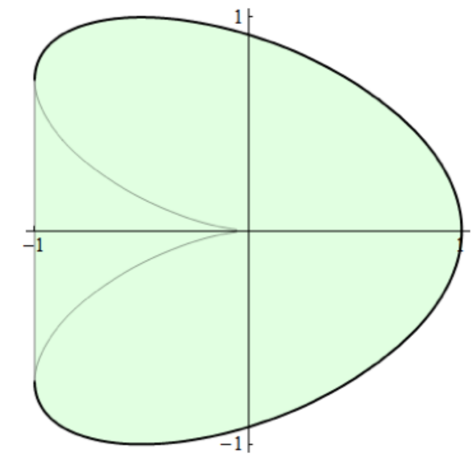
$$S_C \; : \quad \operatorname{ext}(C) \times \operatorname{ext}(C^\circ) \to \mathbf{R}, \qquad (x,y) \mapsto 1 - \langle x, y\rangle$$

$$C^\circ = \{y \,:\, \langle x,y\rangle \leq 1 \;\forall\; x \in C\}$$

$$(1-t^2, 2t-t^3)$$

$$t \in [-\sqrt{2}, \sqrt{2}]$$





$$\left(\frac{2-3s^2}{2-s^2+s^4}, \frac{2s}{2-s^2+s^4}\right)$$

$$s \in [-\sqrt{2}, \sqrt{2}]$$

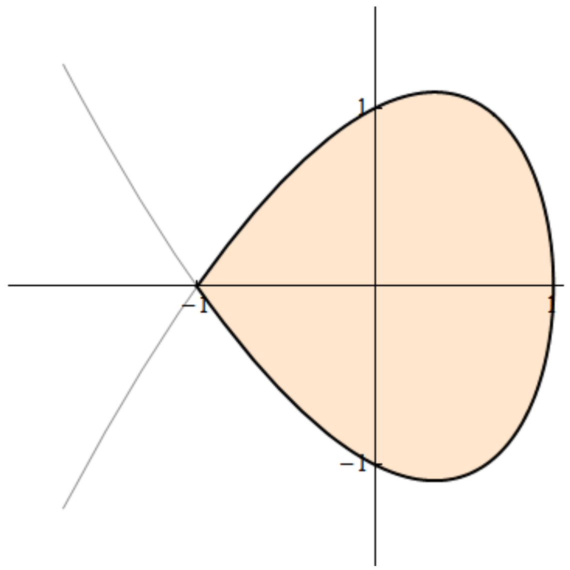
slack operator: $\quad S_C : \operatorname{ext}(C) \times \operatorname{ext}(C^\circ) \rightarrow \mathbf{R}$

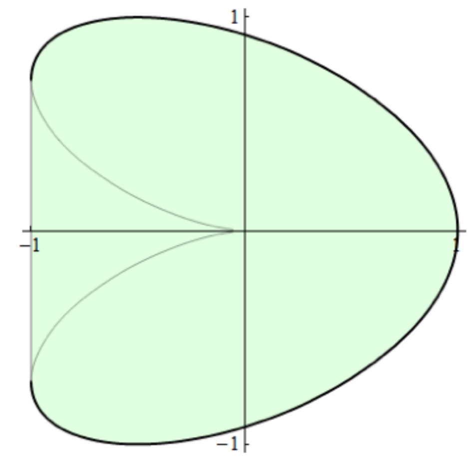
$\qquad\qquad\qquad\qquad\qquad (x, y) \mapsto 1 - \langle x, y \rangle$

$$C^\circ = \{y : \langle x, y \rangle \leq 1 \;\forall\; x \in C\}$$

$(1 - t^2, 2t - t^3)$

$t \in [-\sqrt{2}, \sqrt{2}]$





$\left(\frac{2 - 3s^2}{2 - s^2 + s^4}, \frac{2s}{2 - s^2 + s^4} \right)$

$s \in [-\sqrt{2}, \sqrt{2}]$

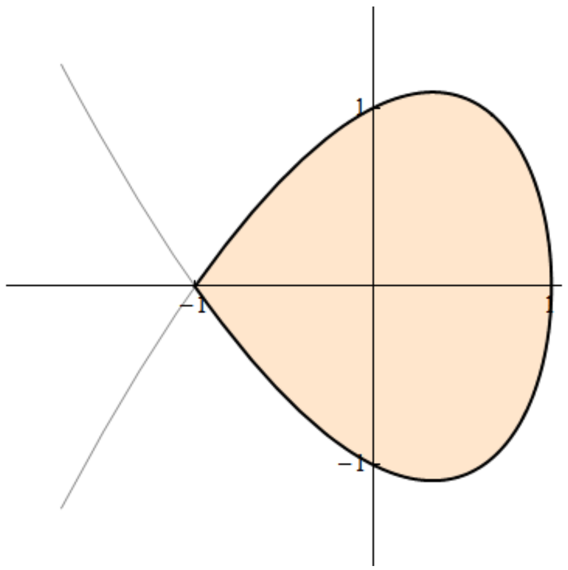
$$s_C : \left[-\sqrt{2}, \sqrt{2}\right] \times \left[-\sqrt{2}, \sqrt{2}\right] \longrightarrow \mathbb{R}_+$$

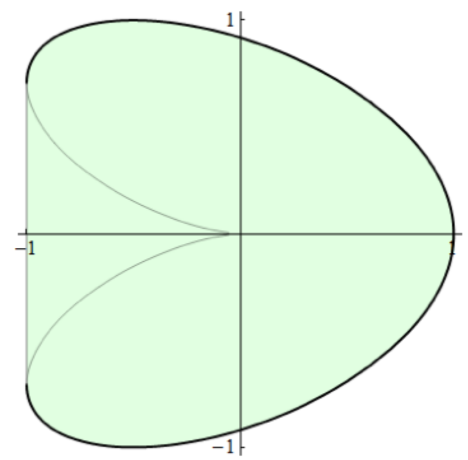
$$s_C(s, t) = 1 - \frac{(2 - 3s^2)(1 - t^2) + 2s(2t - t^3))}{2 - s^2 + s^4}$$

$$= \frac{(2 - t^2)(t - s)^2 + (s^2 - t^2)^2}{s^4 + (2 - s^2)}$$

S_C is K-factorizable if $\exists\ A : \mathrm{ext}(C) \to K, \quad B : \mathrm{ext}(C^\circ) \to K^*$ s.t.

$$S_C(x,y) = 1 - \langle x, y\rangle = \langle A(x), B(y)\rangle$$

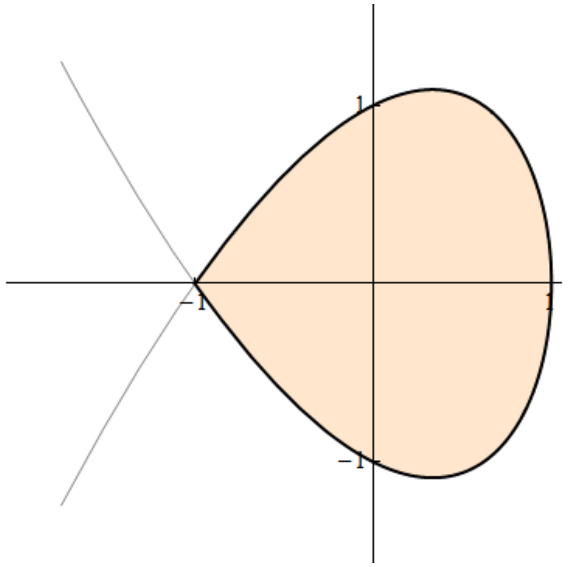




$$S_C \text{ is } K\text{-factorizable if } \exists \;\; A : \operatorname{ext}(C) \to K, \quad B : \operatorname{ext}(C^\circ) \to K^* \text{ s.t.}$$

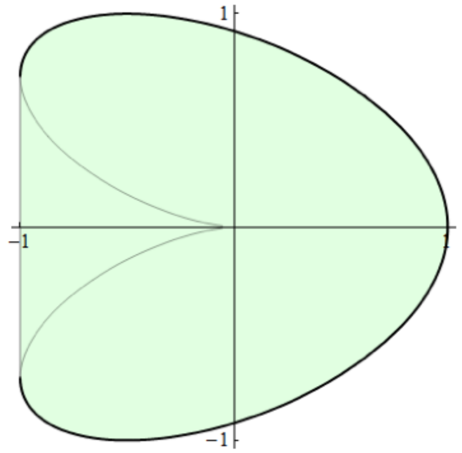
$$S_C(x,y) = 1 - \langle x, y\rangle = \langle A(x), B(y)\rangle$$

$$A(t) = \begin{pmatrix} 1 & 0 & 1-t^2 \\ 0 & 2-t^2 & t(2-t^2) \\ 1-t^2 & t(2-t^2) & 1 \end{pmatrix} \longleftarrow \quad t$$



$$\boxed{\langle A(t), B(s)\rangle = s_C(s,t)}$$

$$B(s) = \frac{1}{2-s^2+s^4} \begin{pmatrix} \left(s^2-1\right)^2 & -s\left(s^2-1\right) & s^2-1 \\ -s\left(s^2-1\right) & s^2 & -s \\ s^2-1 & -s & 1 \end{pmatrix} \longleftarrow \quad s$$

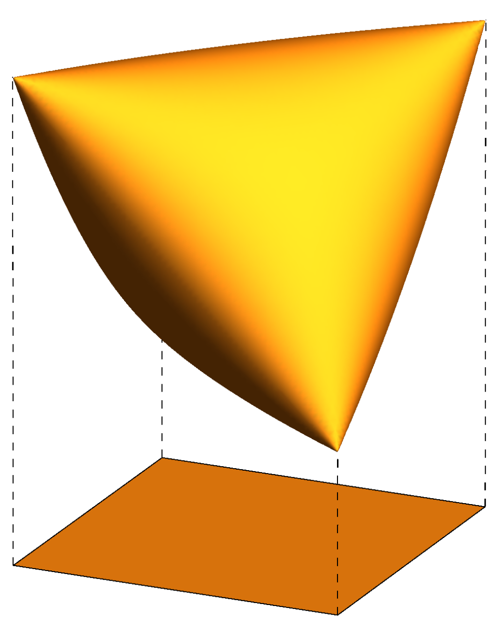


Constructions for Spectrahedral Lifts

$C = \mathrm{conv}(X)$ has $\mathcal{S}_+^m$-lift $\Leftrightarrow$ $\exists$ subspace V of functions on X

$\dim(V) \leq m$ s.t. $\forall\, l(x) \leq 1$ valid on C

$$1 - l(x)|_X = \sum h_k^2 \quad \text{for } h_k \in V$$



$\mathcal{S}_+^3$-lift

$$X = \{(\pm 1, \pm 1)\} = \{(x, y) \,:\, x^2 = 1, y^2 = 1\}$$

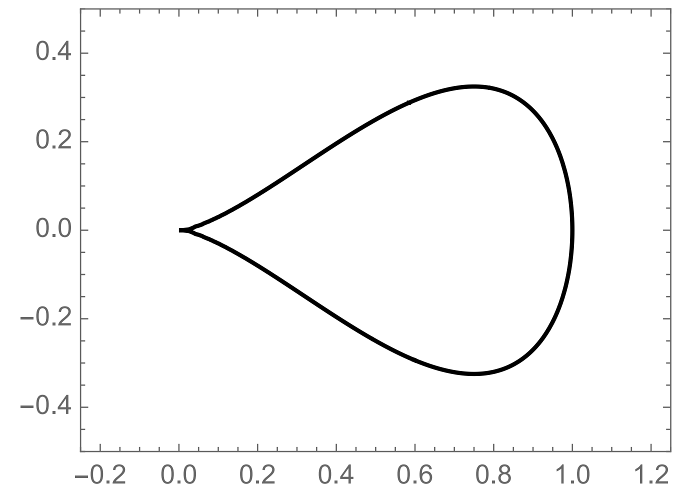
$$C = \mathrm{conv}(X) = [-1, 1]^2$$

$$V = \mathrm{span}(1, x, y)$$

$$1 - x = \frac{1}{2}(1 - x)^2 \;\forall\, x \in X$$

If X algebraic, natural choice for V is all polynomial functions on X of degree at most d

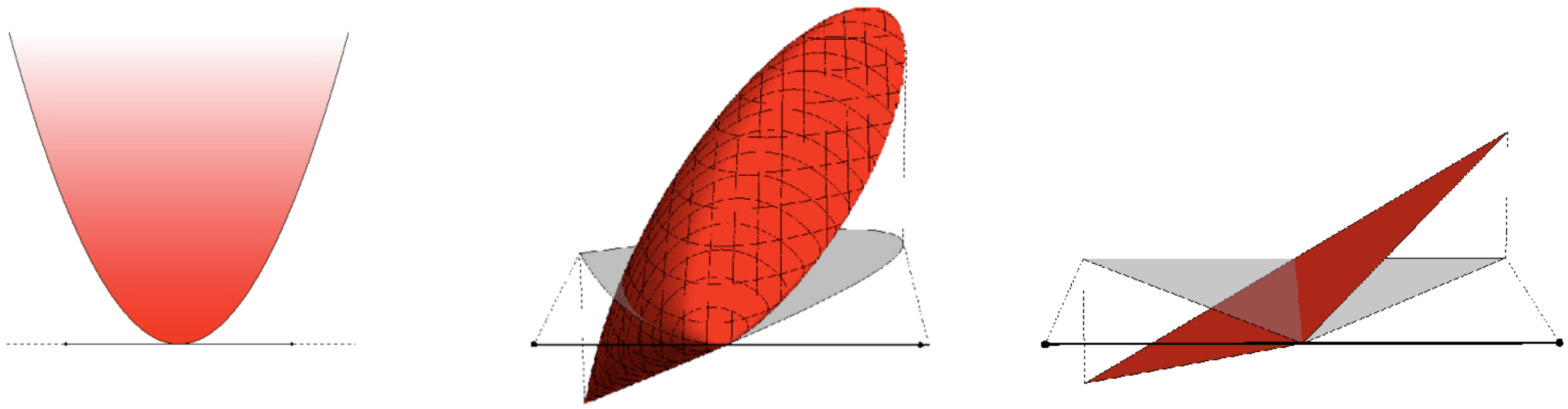
Limitations:



piriform curve

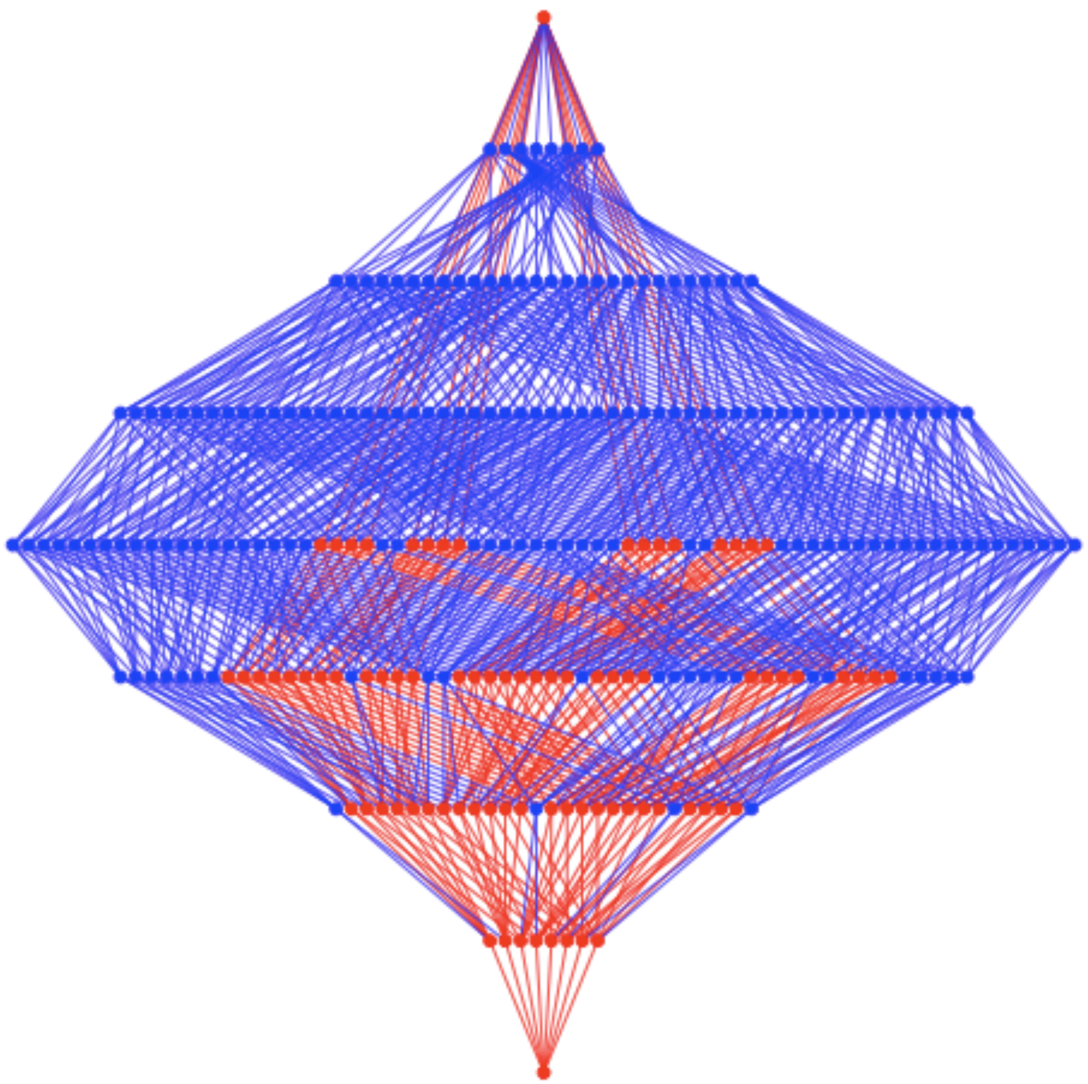
$$y^2 - x^3 + x^4 = 0$$

Heirarchies:



$$I = \langle (x+1)x(x-1)^2 \rangle \qquad \operatorname{conv}(V_{\mathbb{R}}(I)) = [-1, 1]$$

Obstructions & Lower Bounds
(polyhedral lifts)



Kaibel

If Q is a lift of P, the face lattice of P embeds into the face lattice of Q

Goemans (2015)

P polytope, v vertices

$\Rightarrow$ a $\mathbb{R}^m_+$-lift needs $m \geq \lceil \log v \rceil$

$O(n \log n)$-lift of Π_n is optimal

Famous Lower Bounds (polyhedral lifts)

Yannakakis (1991) *symmetric nonnegative rank of the matching polytope of* K_{2n} *is exponential in n*

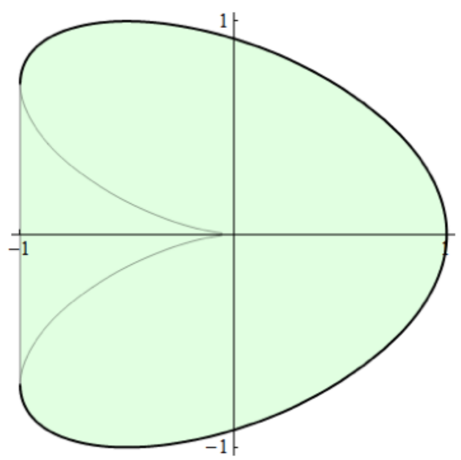
Kaibel-Pashkovich-Theis (2012) *example of a polytope with exponential gap between sizes of symmetric and non-symmetric polyhedral-lift*

Fiorini et al(2015) *traveling salesman polytopes need exponential size polyhedral lifts*

Rothvoss (2013) *polyhedral lift of matching polytope of* K_{2n} *is exponential in n*

Obstructions & Lower Bounds (spectrahedral lifts)

C has a K-lift $\Rightarrow$ length of maximal chain of faces of C $<$ length of maximal chain of faces of K



does not have $\mathcal{S}_+^2$-lift

does not have a smooth cone lift

P polytope, $\dim(P) = n$ has $\mathcal{S}_+^m$-lift $\Rightarrow m \geq n+1$

further obstructions based on neighborliness, facial structure, algebraic properties etc

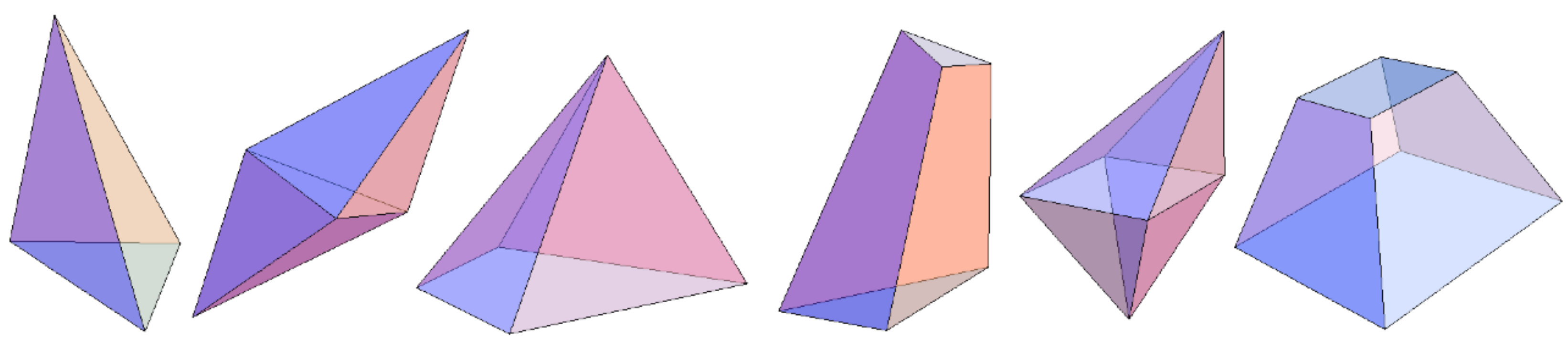
psd-minimal polytopes

Which n-dimensional polytopes have $\mathbf{S}_+^{n+1}$-lifts?

Gouveia-Robinson-T. (2013)

$d = 2:$

$d = 3:$



Theorem: $\mathrm{STAB}(G)$ is psd-minimal $\Leftrightarrow G$ is perfect

Gouveia-Pashkovich-Robinson-T. (2017)

$d = 4:$ *31 combinatorial classes*

Open Questions

- *Is there a family of polytopes with a significant gap between the size of the smallest polyhedral and spectrahedral lifts?*
- *Not much is known about obstructions to lifts for general convex sets.*
- *How does one find subspaces V of functions that guarantee small spectrahedral lifts?*
- *It is known that all convex semi algebraic sets in the plane have spectrahedral lifts (Scheiderer). How small?*
- *Scheiderer also proved that not all convex semi algebraic sets have spectrahedral lifts.*
    - *What is the smallest example?*
    - *Is there one in dim 3?*

Thank You