Contemporary Mathematics

k-Sets and k-Facets

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ABSTRACT. We survey problems, results, and methods concerning k-facets and k-sets of finite point sets in real affine d-space and the dual notion of levels in arrangements of hyperplanes.

CONTENTS

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1. Introduction

Consider a set of n of points in the plane, n even. How many ways are there to partition the points by a straight line into two equal halves? The answer will depend on the particular set of points at hand, as we see when looking at the essentially two different ways of placing four points in the plane, see Figure 1. It is not difficult to

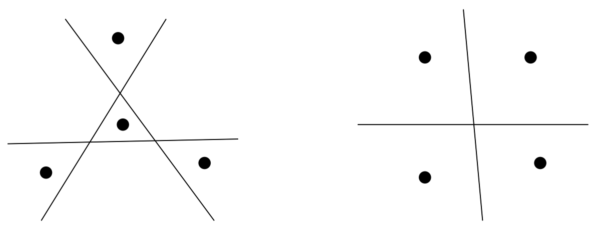


FIGURE 1. Two sets of 4 points in $\mathbf{R}^3$, and their equipartitons by lines.

see that for any set of n points, there always exist at least $n/2$ such partitions (and this minimum is attained, for instance, if the points form the vertices of a convex n-gon). Can we also determine the maximum number of straight-line equipartitions that a set of n points can have? This deceptively simple question has been puzzling discrete and computational geometers for more than thirty years.

More generally, let S be a set of n points in affine space $\mathbf{R}^d$. A subset $A \subseteq S$ is called a k-set of S, where k is an integer parameter, if $|A| = k$ and there is an affine hyperplane h that strictly separates A from its complement $S \setminus A$ (i.e., A is completely contained in one of the open halfspaces determined by h, and $S \setminus A$ in the other). Let us denote the number of k-sets of S by $a_k(S)$. Then the question is: What is the maximum number

$$a_k^{(d)}(n) := \max_{\substack{S \subset \mathbf{R}^d \\ |S|=n}} a_k(S)$$

of k-sets that an n-point set in $\mathbf{R}^d$ can have? This problem is known as the *k-set problem.* The dimension d is usually considered fixed, and the focus is on the asymptotic behavior of the function $a_k^{(d)}(n)$ as $n \to \infty$. Of course, exact answers would be even nicer, but already the order of magnitude seems difficult to determine. Despite considerable efforts by numerous researchers over the last thirty-odd years, and many interesting partial results, the gap between the upper and lower bounds proved to date remains substantial, even in the plane.

1.1. k-Facets.

Instead of free-floating separating hyperplanes, it is often technically more convenient to work with hyperplanes spanned by points in S. A first observation is that a slight perturbation of the points can only increase the number of k-sets.[1] Thus, we may assume without loss of generality that the set S is *in general position* (i.e., that any $d+1$ or fewer of the points are affinely independent).

Consider a *cooriented* $(d-1)$-dimensional simplex σ, spanned by points from S. The affine hull of σ is a hyperplane, which bounds two open halfspaces, and the

[1] For each k-set of S, pick a separating hyperplane disjoint from S. This yields a collection of finitely many (certainly not more than $\binom{n}{k}$) hyperplanes, which decompose $\mathbf{R}^d$ into finitely many regions, and the points of S lie in the interiors of these regions. Therefore, as long as we perturb the points within these regions, we do not affect the original k-sets.

coorientation of σ simply means that one of these two open halfspaces is designated as *positive*, denoted by σ^+, and the other one as *negative*, denoted by σ^-. The simplex σ is called a *k-facet* of S, for integer k, if there are exactly k points from S on the positive side of σ, i.e., $|S \cap \sigma^+| = k$.

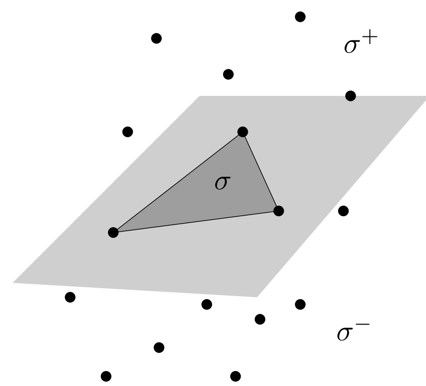


FIGURE 2. Sixteen points in $\mathbf{R}^3$ and a 5-facet.

For example, the 0-facets of S correspond to the facets of $\operatorname{conv}(S)$ (with outer coorientation). In dimension $d = 2$, we often speak of *k-edges* instead of k-facets (and depict them as oriented edges, with the convention that the positive halfplane of a directed edge is to the right of the edge, see Figure 3).

We denote the number of k-facets of S by $e_k(S)$. Observe that $e_k(S) = 0$ for $k \notin \{0, \ldots, n-d\}$, that $e_k(S) = e_{n-d-k}(S)$ (by reversing the coorientation), and that $\sum_k e_k(S) = 2\binom{n}{d}$. We use the notation

$$e_k^{(d)}(n) := \max_{\substack{S \subset \mathbf{R}^d \\ |S|=n}} e_k(S)$$

for the maximum number of k-facets that an n-point set in general position in $\mathbf{R}^d$ can have. When d is clear from the context, we will often drop the superscript and just write $e_k(n)$ and $a_k(n)$. In identities or estimates that hold for all point sets of a given size in a given dimension, we will often simply write e_k or a_k.

It is not hard to see that in the plane $a_k = e_{k-1}$ for $1 \leq k \leq n-1$.[2] One can also show that in three dimensions, $a_k = \frac{1}{2}(e_{k-2} + e_{k-1}) + 2$ for $1 \leq k \leq n-1$ (see Section 2.2), but in higher dimensions, these quantities no longer determine each other (see Andrzejak and Welzl [**21**]). However, it remains true that they are equivalent as far as their order of magnitude is concerned.[3]

[2]More formally, $a_k(S) = e_{k-1}(S)$ for $1 \leq k \leq n-1$ and for all n-point sets S in general position in the plane. To see this, consider a $(k-1)$-edge $\overrightarrow{uv}$; a slight clockwise orientation about the midpoint of the edge creates a directed line with exactly k points (the previous $k-1$, plus v) on the right. Conversely, consider a k-set A and a separating line ℓ, directed so that A lies on its right and $S \setminus A$ on its left. The convex hulls $\operatorname{conv}(A)$ and $\operatorname{conv}(S \setminus A)$ are disjoint. Two disjoint convex bodies in the plane have exactly two common inner bitangents. Exactly one of these two inner bitangents can be obtained from ℓ by a counterclockwise orientation. By general position, this bitangent is spanned by a point in $p \in A$ and a point $q \in S \setminus A$, and then the directed edge $\overrightarrow{pq}$ is a $(k-1)$-edge of S.

[3]It is not hard to see that $a_k \leq \sum_{j=k-d-1}^{k} e_j + O(n^{d-1})$ and $e_j \leq \sum_{k=j}^{j+d-1} a_k$, see, e.g., [**99**, Chapter 11]

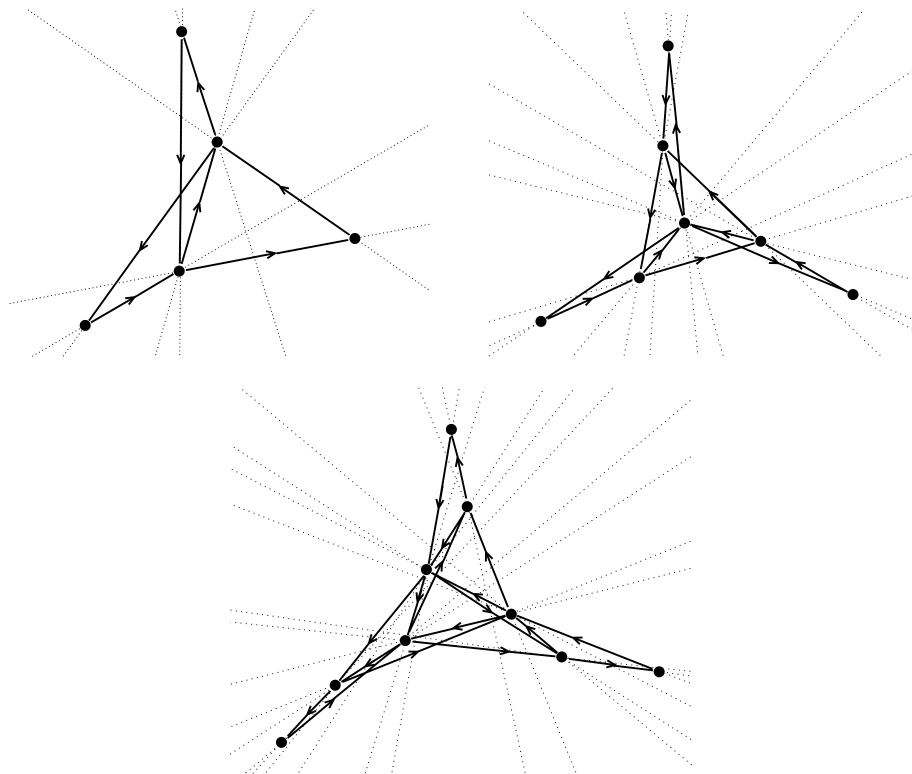


FIGURE 3. Three planar point sets S_n of size $|S_n| = n \in \{5, 7, 9\}$. Each S_n has the maximal number of $\lfloor \frac{n-2}{2} \rfloor$-edges for its size (see Section 9.4).

The question of determining the order of magnitude of $e_k^{(d)}(n)$ was first posed by Simmons (unpublished) in the early 70's, for the special case $d = 2$, n even, and $k = \frac{n-2}{2}$. Straus (also unpublished) found a lower bound of $\Omega(n \log n)$, and Lovász [**95**] proved an upper bound of $O(n^{3/2})$. An extension of this to general k (but still in the plane),

$$\Omega(n \log k) \leq e_k^{(2)}(n) \leq O(n\sqrt{k+1}), \tag{1}$$

appeared, together with Straus' construction, in Erdős, Lovász, Simmons, and Straus [**66**]. They conjectured that in fact,

$$e_k^{(2)}(n) \overset{(?)}{=} O(n(k+1)^{\varepsilon})$$

for every fixed $\varepsilon > 0$.

Halving Facets. It turns out that the case of $k = \lfloor \frac{n-d}{2} \rfloor$ is really the crucial one. More precisely, let us define $e_{1/2} := e_{\lfloor \frac{n-d}{2} \rfloor}$ if $n-d$ is odd, and $e_{1/2} := \frac{1}{2} e_{\frac{n-d}{2}}$ if $n-d$ is even. In the latter case, a *halving simplex* or *halving facet* of S is an *unoriented* $(d-1)$-simplex σ, spanned by points of S, such that there are $\frac{n-d}{2}$ points on either side of the hyperplane $\mathrm{aff}(\sigma)$. It is not hard to see that $e_k^{(d)}(n) \leq 2e_{1/2}(2n-d)$ for all k. What is more, Agarwal, Aronov, Chan, and Sharir [**2**] showed that upper

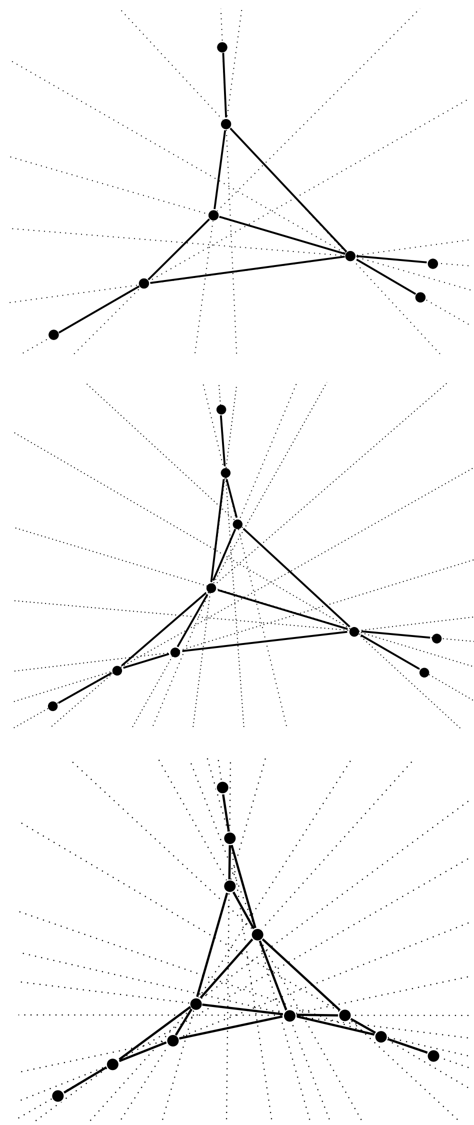


FIGURE 4. Welzl's "little devils": Three sets of eight, ten, and twelve points, respectively, that maximize the number of halving edges (see Section 9.4).

bounds for $e^{(d)}_{1/2}(n)$ yield upper bounds for $e^{(d)}_k(n)$ that are sensitive to k: For any fixed dimension d, if $e^{(d)}_{1/2}(n) = O(n^{d-c_d})$ for some constant c_d, then $e^{(d)}_k(n) = O(n^{\lfloor d/2 \rfloor}(k+1)^{\lceil d/2 \rceil - c_d})$ for all k.

The planar bounds (1) were rediscovered several times (Goodman and Pollack [**72**], Edelsbrunner and Welzl [**60**]); Agarwal, Aronov, Chan, and Sharir [**2**] give an overview of several proof variants for the upper bound. The first improvement of the planar upper bound was achieved by Pach, Steiger, and Szemerédi [**115**], who showed, by a rather involved argument, that $e^{(2)}_{1/2}(n) = O(n^{3/2}/\log^*(n))$.

A real breakthrough was made by Dey [**53**], who analyzed the number of crossings between halving edges. Combining his analysis with the fundamental *Crossing Lemma* due to Ajtai, Chvatal, Newborn, and Szemeredi [**14**] and Leighton [**93**], Dey obtained a very elegant proof of the currently best planar upper bound,

$$e^{(2)}_{1/2}(n) = O(n^{4/3}).$$

We review these 2-dimensional results in Section 7.

In higher dimensions, the first nontrivial upper bound of $e^{(3)}_{1/2}(n) = O(n^{3-c})$, with $c = 1/343$, was proved by Bárány, Füredi, and Lovász [**28**]. The first ingredient of their proof, often called *Lovász' Lemma*, is easy to see to hold in general dimension d: any line intersects at most $O(n^{d-1})$ halving simplices. The second step is to find a line that intersects "a large fraction" of the halving simplices. After a projection onto a generic hyperplane, this reduces to proving a *Point Selection Lemma* to the extent that for any family of "sufficiently many" *full-dimensional* simplices spanned by a set of n points in $\mathbf{R}^d$, there is always a point common to a "large fraction" of the simplices. They key technical ingredient for proving the Selection Lemma is a certain "colorful" generalization of *Tverberg's Theorem*, see Section 2.3. Bárány, Füredi, and Lovász established the *Colored Tverberg Theorem* in the plane, which leads to the three-dimensional bound for halving triangles, and conjectured it to hold in general dimension. This conjecture was proved by Živaljević and Vrećica [**146**] using topological methods. Based on this, Alon, Bárány, Füredi, and Kleitman [**15**] extended the BFL proof to arbitrary dimensions d and showed that

$$e^{(d)}_{1/2}(n) = O(n^{d-\varepsilon_d}),$$

where $\varepsilon_d = (4d-3)^{-d}$ (and the implicit constants depend on d, as usual). We discuss these general upper bounds in Section 6.

For the special cases of three and four dimensions, respectively, the bounds were further improved. First to $e^{(3)}_{1/2}(n) = O(n^{8/3}\log^{5/3} n)$ by Aronov, Chazelle, Edelsbrunner, Guibas, Sharir, and Wenger [**22**], who established an improved 2-dimensional Point Selection Lemma without recourse to the Colored Tverberg Theorem. Dey and Edelsbrunner [**54**] got rid of the logarithmic factor, by abandoning the projection step and arguing directly that for any family of "sufficiently many" triangles in 3-space, there is a line that stabs "many" of them. For this purpose, they work with a generalization of the notion of crossings to pairs of triangles in $\mathbf{R}^3$.

It is worthwhile to note that all present proofs of upper bounds only exploit one simple local property of halving simplices, sometimes called *antipodality* or *interleaving property* (see Section 6.1). Often, this property is only used in one step, for bounding from above the number of occurrences of certain "configurations" (e.g., crossings, or line-simplex intersections) for a family of simplices with the interleaving property. In a second step, it is usually shown that for an *arbitrary* family of "sufficiently" many simplices, such configurations must occur in abundance. The currently best 3-dimensional upper bound of

$$e^{(3)}_{1/2}(n) = O(n^{5/2})$$

was obtained by Sharir, Smorodinsky, and Tardos [**125**], who cleverly exploited the interleaving property also for the second step.

In four dimensions, Matoušek, Sharir, Smorodinsky, and Wagner proved an extension of Lovász' Lemma from lines to planes. Based on this, they showed that

$$e^{(4)}_{1/2}(n) = O(n^{4-2/45}).$$

We discuss the improved bounds in dimensions 3 and 4 in Section 8.

The currently best lower bound of $e^{(2)}_{1/2}(n) = ne^{\Omega(\sqrt{\log n})}$ is due to Tóth [**137**]. By a lifting argument due to Seidel (see [**58**]), this implies a general lower bound of

$$e^{(d)}_{1/2}(n) = n^{d-1}e^{\Omega(\sqrt{\log n})}$$

for any fixed dimension d. We briefly review the lower bounds in Section 5.

1.2. Levels in Arrangements of Hemispheres and Halfspaces. Many people like to think of the problem of k-facets in the dual setting of levels in arrangements. Let $\mathcal{A}$ be a set of n closed affine halfspaces in $\mathbf{R}^d$, which we can think of as the constraints of a linear program. The *level* of a point $x \in \mathbf{R}^d$ with respect to $\mathcal{A}$ is defined as the number of constraints that x violates, i.e., the number of halfspaces that it is *not* contained in. The hyperplanes bounding the halfspaces define a decomposition of $\mathbf{R}^d$ into convex polyhedral cells, or *faces*, of dimensions $i = 0, 1, \dots, d$, see Figure 5. If we want to stress this decomposition, we also speak of the *arrangement* $\mathcal{A}$. We will assume throughout that the arrangement is *simple*, i.e., that any d of the bounding hyperplanes intersect in exactly one point and that no $d+1$ of them have a point in common. Two points that lie in the same face of the arrangement have the same level, so it makes sense to speak of the level of a face.

The boundary of the set $\{x \in \mathbf{R}^d : \text{level}(x) \leq k\}$ of points at level not exceeding k is called *the k-level* of the arrangement. It is a hypersurface that consists of the $(d-1)$-dimensional faces at level k, (some, but not all) $(d-2)$-dimensional faces at level k or $k-1$, and so forth, and of (some, but not all) vertices at levels $k-d+1, \dots, k$, see Figure 5. If the arrangement is *feasible*, i.e., if the intersection of all halfspaces is nonempty, then the k-level is either empty or a topological sphere or disk of dimension $d-1$ (depending on whether it is bounded or unbounded). If the arrangement is infeasible, the k-level may consist of several components, see Figure 6.

Let $v_k(\mathcal{A})$ denote the number of *vertices* (i.e., 0-dimensional faces) at level k in the arrangement $\mathcal{A}$. It is not hard to see that for any fixed dimension d, the number of faces (of any dimension) at level k is at most $\sum_{\ell=k-d+1}^{k} v_\ell(\mathcal{A}) + O(n^{d-1})$ (see Section 10.3). Thus, to determine the order of magnitude of any of these face numbers and the complexity of the k-level, it is sufficient to study the numbers $v_k(\mathcal{A})$.

The definitions carry over verbatim to arrangements of great hemispheres in the d-dimensional sphere $\mathbf{S}^d$. In such a spherical arrangement, the faces come in antipodal pairs, and if a face F has level ℓ and lies at the intersection of i bounding great $(d-1)$-spheres (in simple arrangements, these are precisely the $(d-i)$-dimensional faces), then the antipodal face $-F$ has level $n-i-\ell$. In particular, $v_k = v_{n-d-k}$ for spherical arrangements. By homogenization (i.e., by embedding $\mathbf{R}^d$ as the affine hyperplane $\{x_{d+1} = 1\}$ into $\mathbf{R}^{d+1}$), an affine arrangement of n

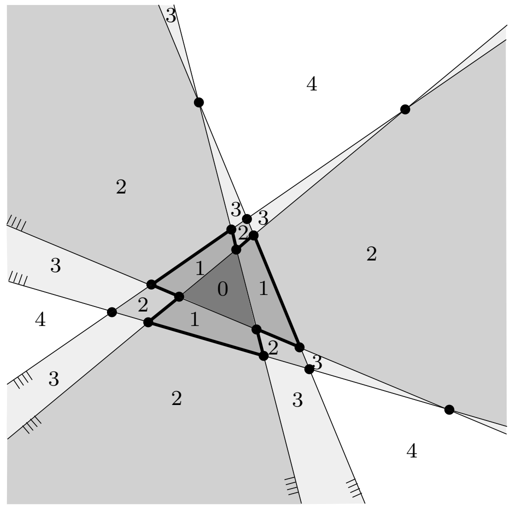


FIGURE 5. Six halfplanes in $\mathbf{R}^2$ (indicated by the little combs attached to the bounding lines). The numbers (and the shading) indicate the levels of the 2-dimensional faces. The level of a lower-dimensional face equals the smallest level of an incident 2-dimensional face. The 1-level is drawn in bold.

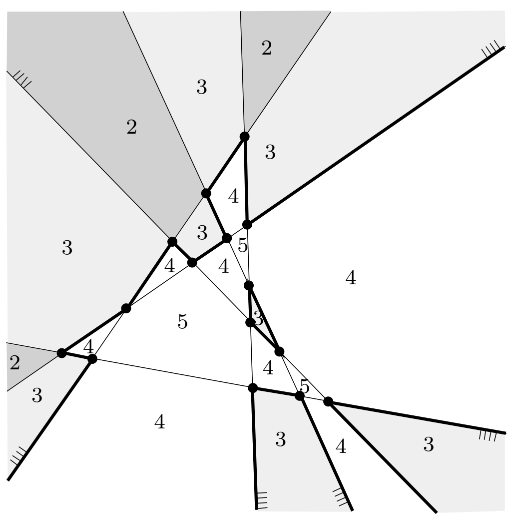


FIGURE 6. The 3-level in an infeasible arrangement of six halfplanes.

halfspaces in $\mathbf{R}^d$ corresponds bijectively to an arrangement of n hemispheres in $\mathbf{S}^d$ together with an additional distinguished "northern hemisphere" (which is not considered to be part of the arrangement), see Figure 7.

The notions of levels and k-facets are closely related by *polar duality* (see Section 2.1). To every feasible arrangement $\mathcal{A}$ of n hemispheres in $\mathbf{S}^d$, there corresponds a set S of n points in $\mathbf{R}^d$, and vice versa, such that the k-facets of S are

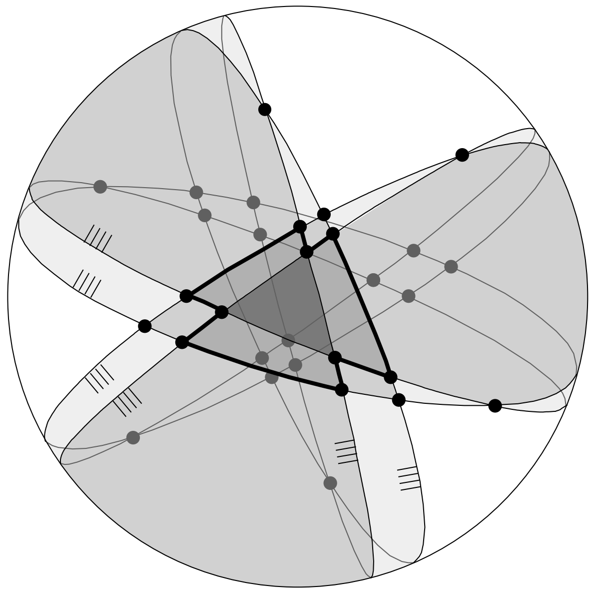


FIGURE 7. The spherical arrangement corresponding to the affine one in Figure 5. In this figure, the distinguished "northern hemisphere" is the hemisphere facing the reader.

in one-to-one correspondence with the vertices of $\mathcal{A}$ at level k.[4] Under duality, the choice of a "northern hemisphere" corresponds to choosing a distinguished "origin" o in $\mathbf{R}^d$. Thus, a set S of n points together with an "origin" o corresponds to a feasible arrangement of affine halfspaces, and the vertices at level k in the affine arrangement correspond to those k-facets of the dual point set S that contain o on their negative side.

In particular, if we chose a a direction in $\mathbf{R}^d$, say the x_d-axis, which we think of as "vertical", and place the origin at "$x_d = +\infty$", then we obtain a standard point-hyperplane duality transform under which a set S of n points in $\mathbf{R}^d$ corresponds to n non-vertical affine hyperplanes, where we implicitly associate the lower halfspace with each hyperplane. In this setting, a point is at level k in the arrangement if there are exactly k hyperplanes that pass below it, and a vertex at level k in the arrangement corresponds to a "lower k-facet" (i.e., there are k points below the simplex, where "upper" and "lower" and "above" and "below" refer to the chosen vertical direction). In this sense, one often speaks of levels in arrangements of hyperplanes.

If we allow also infeasible arrangements, there are no nontrivial upper bounds on the complexity of a single level. For example, in a d-dimensional arrangement that is Gale dual to a neighborly polytope, all vertices lie at the middle levels $\lfloor \frac{n-d}{2} \rfloor, \lceil \frac{n-d}{2} \rceil$, see Corollary 2.7.

[4]If the intersection $\bigcap \mathcal{A}$ is empty, then we get a "signed" point configuration, where each point carries a sign ± 1.

1.3. $(\leq k)$-Facets and $(\leq k)$-Levels. For many applications, it is possible to replace upper bounds for the complexity of a single level by the bounds for the combined complexity of the first k levels, i.e., to replace v_k by $v_{\leq k} := \sum_{j\leq k} v_j$ (and dually, e_k by $e_{\leq k} := \sum_{j\leq k} e_j$). Moreover, these quantities are much better behaved and understood. The $(\leq k)$-level also arises naturally when studying a relaxed version of linear programming, where a limited number of constraints are allowed to be violated [**98, 44**]. Unlike in the case of a single level, nontrivial upper bounds for the $(\leq k)$-level can be obtained for both feasible and infeasible arrangements, and essentially all available methods apply to both cases alike, so there is no need to restrict one's attention. The extremal examples seem to be feasbile, but even when studying only these, infeasible arrangements occur naturally as subproblems. Clarkson and Shor [**50**] proved upper bounds for the complexity of the $(\leq k)$-level in all dimensions: For any arrangement $\mathcal{A}$ of n hemispheres in $\mathbf{S}^d$,

$$v_{\leq k}(\mathcal{A}) \leq 2\Big(\frac{e}{\lceil d/2\rceil}\Big)^{\lceil d/2\rceil}\binom{n}{\lfloor d/2\rfloor}(k+\lceil d/2\rceil)^{\lceil d/2\rceil}.$$

For any fixed dimension d, this is $O(n^{\lfloor d/2\rfloor}(k+1)^{\lceil d/2\rceil})$, and this bound is tight up to constant factors (by Proposition 4.1), but it assumes its full strength only if n is large compared to d.

By definition, the points at level 0 in an arrangement form a convex polyhedron, and this case $k = 0$ is the only one for which exact bounds are known for all values of n and d. This is the content of McMullen's Upper Bound Theorem for convex polytopes [**102**], which includes a characterization of when the bounds are attained, see Section 10.2. The Upper Bound Theorem also provides the base case for the general random sampling technique that Clarkson and Shor use to obtain their bound for the $(\leq k)$-level.

Eckhoff [**56**], Linhart [**94**], and Welzl [**145**], independently of one another, conjectured a generalization of the Upper Bound Theorem to levels in arrangements, namely an exact upper bound for the $(\leq k)$-level in an arrangement of hemispheres for all n, d, and $k \leq (n-d-1)/2$. Apart from its intrinsic interest as a sharpening of the Clarkson-Shor bound, such an exact bound would have interesting connections to the so-called *Generalized Lower Bound Theorem* in polytope theory, and to crossing numbers of complete graphs, see Section 10.

The conjecture is known to be true in full generality in dimension 2 (Peck [**117**], Alon and Győri [**16**]), and for feasible arrangements in dimension 3 (Welzl [**145**]). A weaker form of the conjecture for arrangements of halfspaces was proved by Linhart [**94**] for dimension $d \leq 4$. Recently, the weak form of the conjecture was proved [**142**], up to a factor of 2, for all values of n, d, and k, which implies the strong conjecture up to a factor of 4.

1.4. Arrangements of Other Hypersurfaces. When studying the complexity of levels in arrangements of halfspaces or hemispheres, it is natural also to consider the generalization of the problem to arrangements of other kinds of (cooriented) hypersurfaces or hypersurface patches in $\mathbf{R}^d$ or in $\mathbf{S}^d$. An immediate generalization are arrangements of so-called *pseudo(hemi)spheres*, or *oriented matroids*, which are a combinatorial abstraction of arrangements of (hemi)spheres. The book by Björner, Las Vergnas, Sturmfels, White, and Ziegler [**32**] is the standard reference on oriented matroids.

For more general hypersurfaces and hypersurface patches, one usually assumes that they are semialgebraic of bounded description complexity (i.e., defined by a bounded number of polynomial equations and inequalities of bounded degrees) and in general position (for instance, the intersection of any d hypersurface should locally look like the intersecion of d hyperplanes), which can always be ensured by assuming that the coefficients of the defining polynomials are algebraically independent.

Furthermore, it is usually assumed that the relative interior of each hypersurface is smooth and that each hypersurface is either closed and encloses a bounded region (like in the case of spheres or ellipsoids), or that the hyerpsurface is $(x_1, \ldots, x_{d-1})$-*monotone*. The latter means that each hypersurface looks like the graph of a (partial) function from $\mathbf{R}^{d-1} \to \mathbf{R}^d$ (which can be ensured by stratification algorithms, see, e.g., Bochnak, Coste, and Roy [**34**]). These assumptions yield a straightforward coorientation. In the first case, the level of a point is defined as the number of enclosed regions that it is contained in; in the second case, the level of a point is the number of hypersurfaces (or patches) below it.

In this survey, we will focus on most basic linear case of arrangements of half-spaces and hemispheres, respectively of points and k-facets in the primal setting. Here, we just summarize briefly what is known for more general arrangements.

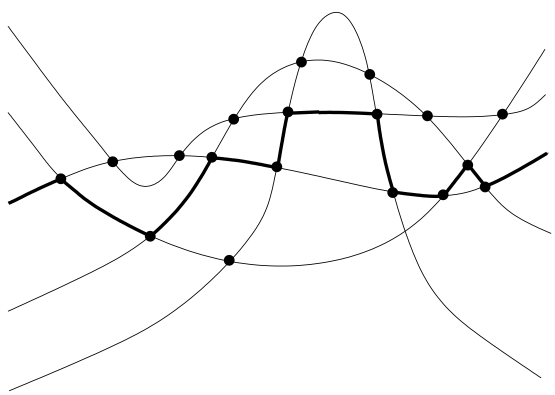


FIGURE 8. The 2-level in an arrangement of 5 pseudoparabolas.

Particular attention has been given to arrangments of line segments and to *arrangements of s-intersecting curves* in the plane; the latter means collections of graphs of continuous functions $\mathbf{R} \to \mathbf{R}$ such that any two intersect at most s times, for some constant s. In the special cases $s = 1, 2$, one speaks of arrangements of *pseudolines* and *pseudoparabolas*, respectively, see Figure 8.

All known bounds for the number of k-edges in the plane (or levels in arrangements of lines) carry over to levels in in pseudoline arrangements. Some proofs [**72, 60**] were originally formulated in this set-up. Dey's proof was generalized to pseudolines by Tamaki and Tokuyama [**135**]; this was further simplified Sharir and Smorodinsky [**124**].

Tamaki and Tokuyama [**134**] proved an upper bound of $O(n^{23/12})$ for the complexity of any level in an arrangement of pseudoparabolas. This was improved by Agarwal, Nevo, Pach, Pinchasi, Sharir, and Smorodinsky [**4**] and further by

Chan [**43, 42**]. The best bound is $O(n^{3/2}\operatorname{polylog}(n))$. More generally, for any constant s, Chan's method yields upper bounds for the complexity of a single level in arrangements of s-intersecting curves, namely $O(n^{2-\frac{1}{2s}})$ for odd $s \geq 3$ and $O(n^{2-\frac{1}{2(s-1)}})$ for even $s \geq 4$.

Agarwal, Aronov, Chan, and Sharir [**2**] considered k-levels in arrangements of line segments in the plane and of triangles in $\mathbf{R}^3$. For the k-level in an arrangement of segments, their method gives an upper bound of $\alpha(\frac{n}{k+1})$ times the bound for line arrangements, where α is the extremely slowly growing inverse Ackermann function. Together with Dey's result, this implies an upper bound of $O(n(k+1)^{1/3}\alpha(\frac{n}{k+1}))$. For the k-level in an arrangement of n triangles in $\mathbf{R}^3$, Agarwal et al. proved an upper bound of $O(n^2(k+1)^{5/6}\alpha(n/(k+1)))$; this was improved to $O(n^2k^{2/3}$ by Katoh and Tokuyama [**86**].

1.5. Computations and Applications. The notion of k-sets and its variants naturally occur in many contexts.

1.5.1. *Order-k Voronoi diagrams.* One example are nearest-neighbor problems in Euclidean space. Let S be a set of n points in $\mathbf{R}^d$, called "sites", and let k be an integer parameter, $1 \leq k \leq n-1$. We would like a data structure that, given a query point $q \in \mathbf{R}^d$, allows us to determine the k sites in S that are nearest (with respect to the Euclidean distance) to q. The order-k Voronoi diagram $\mathcal{V}_k(S)$ is a polyhedral cell complex such that the interiors of the cells form a partition of $\mathbf{R}^d$, see Figure 9. Each d-dimensional cell corresponds to a k-element subset

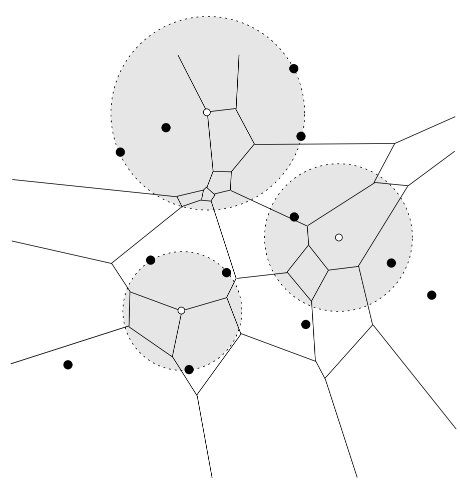


FIGURE 9. The order-2 Voronoi diagram of twelve points (drawn in black) in the plane. For a point x that lies in the interior of a cell, there is a disk centered at x that contains precisely two sites, the two sites cloesest to x. Moreover, y lies in the same cell if it has the same two closest sites. For every vertex v of the diagram, there is a disk centered at v and spanned by three sites that contains at most one site in its interior.

$A \subset S$ and all the points for which the sites in A are the k nearest ones. More precisely, for $A \subseteq S$ and $q \in \mathbf{R}^d$, define $\operatorname{dist}(q, A) := \max_{a \in A} \|q - a\|_2$. Then the cell corresponding to $A \in \binom{S}{k}$ is

$$C(A) = \{q \in \mathbf{R}^d : \operatorname{dist}(q, A) \leq \operatorname{dist}(q, B) \text{ for all } B \in \tbinom{S}{k}\}.$$

In other words, $c \in C(A)$ iff there exists a Euclidean ball centered at c that contains A and is disjoint from $S \setminus A$. The order-k Voronoi diagram $\mathcal{V}_k(S)$ consists of all the nonempty cells $C(A)$, $A \in \binom{S}{k}$, and their lower-dimensional faces.

Order-k Voronoi diagrams in $\mathbf{R}^d$ correspond to k-sets in $\mathbf{R}^{d+1}$ via a *paraboidal lifting map.* Let $U = \{x \in \mathbf{R}^{d+1} : x_{d+1} = \sum_{i=1}^d x_i^2\}$ be the unit paraboloid. If we identify $\mathbf{R}^d = \mathbf{R}^d \times \{0\} \subset \mathbf{R}^{d+1}$ and lift every point $p \in S$ vertically to the point $\hat{p} = (p_1, \ldots, p_d, \sum_i p_i^2)$ on U, then a set $A \in \binom{S}{k}$ determines a nonempty order-k Voronoi cell of S iff the lifting $\hat{A}$ is a lower k-set of $\hat{S}$, because a separating ball lifts to a separating lower halfspace.[5]

If we apply vertical point-hyperplane duality in $\mathbf{R}^{d+1}$, then the lifted set $\hat{S}$ corresponds to the arrangement $\mathcal{A}$ of tangent hyperplanes $h_p = \{x \in \mathbf{R}^{d+1} : x_{d+1} = 2\sum_{i=1}^d p_i x_i - \sum_{i=1}^d p_i^2\}$ at U, and the cells $C(A)$ in the order-k Voronoi diagram of S are the vertical projections of the $(d+1)$-dimensional faces of $\mathcal{A}$ at upper level k (i.e., such that exactly k hyperplanes pass above it), see Figure 10.

1.5.2. *Randomized incremental convex hull computation.* Another classical application of k-facets is the analysis of certain convex hull algorithms. Suppose we wish to compute the convex hull of a finite set S of points in d-dimensional Euclidean space $\mathbf{R}^d$. Of course, we have to say what we mean by "computing the convex hull", i.e., to specify the desired output. For simplicity, let us assume that the points are in general position. Then the boundary of the convex hull (with outer coorientation) consists precisely of the 0-facets of S. Thus, a possible description of $\operatorname{conv}(S)$ that we might want to compute would be a list of these 0-facets (each of which can be represented by listing the d points of S that form its vertices.).

A natural approach to this problem is an incremental algorithm: Enumerate the points as $S = \{p_1, p_2, p_3, \ldots, p_n\}$ and add the points one by one. If we have already computed $C_i = \operatorname{conv}(\{p_1, \ldots, p_i\})$, we first check which 0-facets of C_i are "destroyed" by inserting p_{i+1}: these are precisely the 0-facets σ of C_i with $p_{i+1} \in \sigma^+$. Of course, one also has to compute which new facets incident to p_{i+1} arise, and in order to do these update computations efficiently, one needs a richer description of the convex hull and various auxiliary data structures, which we will not discuss here.

In any case, the performance of the algorithm depends strongly on the order in which the points are inserted. For instance, in dimension $d = 3$, the convex hull of n points has at most $2n - 4$ facets, but it is easy to conceive malicious insertion orders that cause $\Theta(n^2)$ facets to be created (and destroyed again) in the course of the algorithm, see Figure 11.

A standard remedy for this kind of problem is to choose an insertion order uniformly at random. If we do this, we can analyze the expected number of facets

[5]To see this, suppose that the cell $C(A)$ is nonempty and consider a point q in its interior. This means that there is a ball of some radius r centered at q that contains A in its interior and is disjoint from $S \setminus A$. In other words, $\|a - c\|_2^2 < r < \|p - c\|_2^2$ for all $a \in A$ and all $a \in A$ and all $p \in S \setminus A$. When we expand the squared norms in terms of the coordinates, we see that $c_1^2 + \ldots + c_d^2$ appears on both sides. Subtracting this term, we see that $\sum_{i=1}^d 2a_i c_i + a_i^2 < r - \sum_{i=1}^d c_i^2 < \sum_{i=1}^d 2p_i c_i + p_i^2$. Thus, the hyperplane $h = \{x \in \mathbf{R}^{d+1} : 2\sum c_i x_i + x_{d+1}\}$ has $\hat{A}$ below it and $\hat{S} \setminus \hat{A}$ above it.

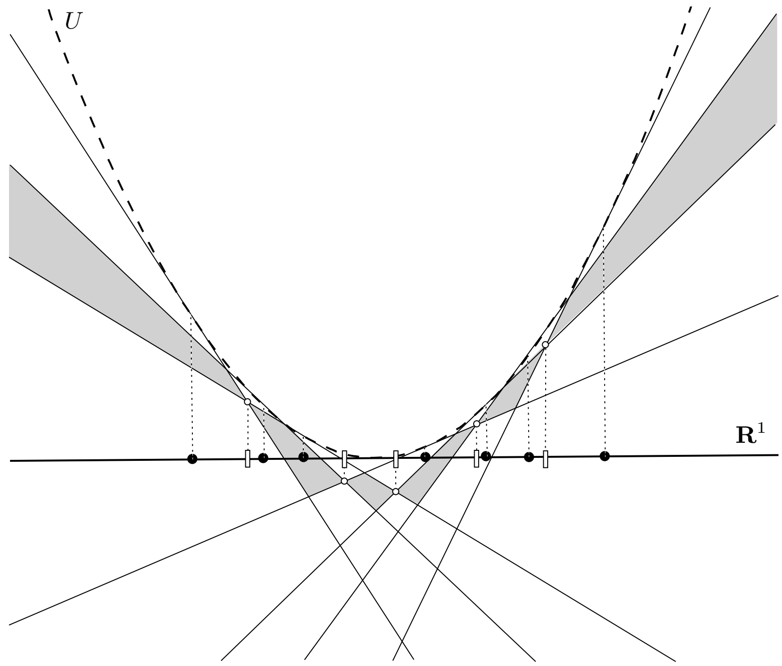


FIGURE 10. The order-2 Voronoi diagram of seven points in $\mathbf{R}^1$ (drawn as black dots) as the projection of the full-dimensional faces at upper level 2-level of the arrangement of tangents to the unit paraboloid U in $\mathbf{R}^2$; the small white vertical line segments indicate the boundaries of the Voronoi cells. A point in the interior of a full-dimensional face corresponds to a disk that contains exactly two of the original points.

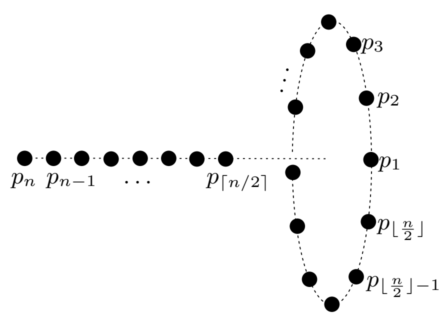


FIGURE 11. A bad insertion order.

that appear during the algorithm as follows. Consider a $(d-1)$-dimensional simplex σ, and for convenience, also fix a coorientation. Then σ arises as a 0-facet of some intermediate C_i if and only if the d points defining σ are inserted before any of the points in the open halfspace σ^+. If there are k such points, i.e., if σ is a k-facet, then the probability for this to occur is $1/\binom{d+k}{d}$. By summing up over all k, the expected number of $(d-1)$-dimenional simplices that appear as facets that appear at some point of the computation equals

$$\sum_k \frac{1}{\binom{d+k}{d}} e_k(S).$$

We do not have good upper bounds for the numbers e_k, but by summation by parts, one can rewrite the sum in terms of the numbers $e_{\leq k}$ and thus obtains

$$2 \ + \sum_{k<(n-d)/2} \frac{1}{\binom{d+k}{k}} \left(1 - \tfrac{k+1}{d+k+1}\right) e_{\leq k}.$$

Substituting the tight upper bounds for the numbers $e_{\leq k}$, one can show that the sum evaluates to $O(n^{\lfloor d/2 \rfloor})$ for fixed d. In general, this is best possible, because up to a constant factor, the final convex hull can already have that many facets. Similar sums appear in the analysis of the running time of the algorithm, see Clarkson and Shor [**50**].

Further applications of k-sets include parametric matroid optimization (see Eppstein [**64**] and Katoh, Tamaki, and Tokuyama [**85**]) and orthogonal L_1-hyperplane fitting. In the latter problem, we are given a set S of n points in $\mathbf{R}^d$, and we are looking for a hyperplane h that minizmizes the sum of vertical distances from the points to h (we refer the reader to Korneenko and Martini [**90**] for a survey of this and other hyperplane approximation questions). By considering parallel translates, one sees that an optimal hyperplane must pass through at least one point in S and split the remaining points as evenly as possible. Thus, in the dual arrangement, the optimum corresponds to a point on the middle level.

1.5.3. *Enumerating k-Sets and Constructing Levels.* Mulmuley [**106**] gave a randomized incremental algorithm algorithm for computing the $(\leq k)$-level in hyperplane arrangements in any dimension. For $d \geq 4$, the expected running time of this algorithm is $O(n^{\lfloor d/2 \rfloor}(k+1)^{\lceil d/2 \rceil})$, which is optimal in the worst case. For $d = 2, 3$, the expected running time exceeds the worst-case complexity of the $(\leq k)$-level by a logarithmic factor of $\log(n/(k+1))$. Everett, Robert, and van Kreveld [**68**] presented an algorithm for computing the $(\leq k)$-level in the plane with expected running time $O(n \log n + nk)$, which is worst-case optimal. Agarwal, de Berg, Matoušek, and Schwarzkopf [**3**] gave an algorithm for computing the $(\leq k)$-level of lines in the plane or of planes in $\mathbf{R}^3$. The worst-case running time in three dimensions is $O(nk^2 + (n \log n)^3)$. Sharir [**122**] gave an algorithm to compute levels in fairly general arrangements in the plane; the worst-case running time is roughly $(\log n)^2$ times the maximum complexity of the $(\leq k)$-level.

For single levels, Edelsbrunner and Welzl [**61**] presented an output-sensitive algorithm for computing the k-level of n lines in the plane; the running time was further improved to $O(n \log n + (\log k)^2 m)$ by Cole, Sharir, and Yap [**51**], where m is the actual complexity of the k-level in the input arrangement. Chan [**39**] observed that using new data structures for the dynamic planar convex hull problem [**40, 36**], the exponent in the logarithmic overhead can be brought down to 1. Chan also presented a randomized algorithm that computes the k-level in expected time $O(n \log n + k^{1/3} n)$. If Dey's upper bound should turn out to be tight, then this would be an optimal algorithm.

In higher dimensions, many algorithms for computing the k-level of n hyperplanes actually compute the entire $(\leq k)$-level, or have a similar time complexity.

Special attention has been paid to the problem of computing the k-level of hyperplanes tangent to the unit paraboloid in $\mathbf{R}^d$, which, as we have seen, is equivalent to computing the k-th order Voronoi diagram of n points in $\mathbf{R}^{d-1}$ [**106, 45, 23**]. For $d = 3$, the algorithm of Agarwal, de Berg, Matoušek, and Schwarzkopf [**3**] solves this in $O(k(n-k)\log n + n(\log n)^3)$ expected time, which is worst-case optimal up to logarithmic factors.

Andrzejak and Fukuda [**20**] presented a different set of algorithms based on the reverse search technique by Avis and Fukuda [**24**]. These algorithms have certain practical advantages, but the theoretical worst case upper bounds for the running time are worse than for previous algorithms.

2. Preliminaries

2.1. Polar Duality. Polar duality on the unit sphere is just the corresponence between a unit vector $u \in \mathbf{S}^d$ and the hemisphere with outer normal vector u,

$$u \quad \leftrightarrow \quad H = \{x \in \mathbf{S}^d : \langle u, x\rangle \leq 0\}.$$

If H and u are related in this way, we write $H = u^*$ and $u = H^*$.

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a simple arrangement of n hemispheres in $\mathbf{S}^d$, and let $U = \{u_1, \ldots, u_n\} \subset \mathbf{S}^d$ be the collection of the corresponding outer normal vectors. Consider a point $v \in \mathbf{S}^d$, and let $C = \{i : v \notin H_i\}$ be the set of (indices of) hemispheres contributing to the level of v and $B = \{i : v \in \partial H_i\}$ the set of hemispheres whose bounding $(d-1)$-spheres pass through v. Then v^* is a hemisphere that contains precisely the points $\{u_i : i \in C\}$ in its interior and whose boundary passes through $\{u_i : i \in B\}$. The sets B and C are determined by the face of the arrangement in whose relative interior v lies.

In particular, if is v is a vertex level k, then the boundary of v^* is spanned by the d points u_i, $i \in B$, and we could call the spherical convex hull of these spanning vectors (with the coorientation given by the hemisphere v^*) a *spherical k-facet* of the vector configuration U. On the other hand, if v lies in the interior of a d-dimensional face of the arrangement, then $B = 0$ and $\{u_i : i \in C\}$ is a *spherical k-set* of U.

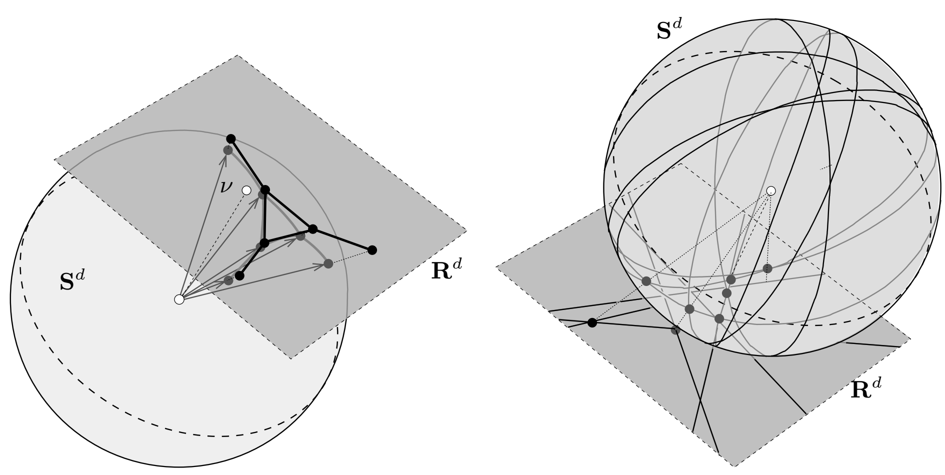


FIGURE 12. A sketch of polar duality.

$S = \{p_1, \ldots, p_n\}$ (n points in $\mathbf{R}^d$)	$\leftrightarrow$	$\mathcal{A} = \{H_1, \ldots, H_n\}$ (n hemispheres in $\mathbf{S}^d$), with $\bigcap_{i=1}^n H_i \neq \emptyset$
halfspace H in $\mathbf{R}^d$, $B = \{i : p_i \in \partial H\}$, $C = \{i : p_i \notin H\}$	$\leftrightarrow$	point $v \in \mathbf{S}^d$, $B = \{i : v \in \partial H_i\}$, $C = \{i : v \notin H_i\}$
k-facet σ of S	$\leftrightarrow$	vertex v of $\mathcal{A}$ at level k

TABLE 1. Dictionary for polar point-hemisphere duality.

2.1.1. *(De)homogenization and polar duality in affine space.* We can interpret $\mathbf{R}^d$ as a tangent hyperplane to $\mathbf{S}^d$ and apply a radial projection from the center of the sphere. Then an arrangement of n affine halfspaces in $\mathbf{R}^d$, together with the choice of an "origin" in $\mathbf{R}^d$ (the point of tangency in $\mathbf{R}^d$), corresponds to an arrangement of n hemispheres in $\mathbf{S}^d$, together with a "north pole" ν (the point of tangency in $\mathbf{S}^d$). More precisely, the affine arrangement corresponds bijectively to that part of the spherical arrangement that lies in the "northern hemisphere" $\{x \in \mathbf{S}^d : \langle \nu, x\rangle > 0\}$ (this distinguished hemisphere itself is not considered to be in the arrangement).

Similarly, a set $S = \{p_1, \ldots, p_n\}$ of n points in $\mathbf{R}^d$, together with a distinguished origin, corresponds to a set $U = \{u_1, \ldots, u_n\}$ of n vectors in $\mathbf{S}^d$, together with an open hemisphere that enirely contains U, by radial projection onto the tangent hyperplane at the "pole" of that hemisphere.[6] The spherical k-facets [spherical k-sets] of U correspond to the k-facets [k-sets] of S.

Furthermore, a set $U = \{u_1, \ldots, u_n\} \subset \mathbf{S}^d$ of vectors has the property that it is entirely contained in a hemisphere (which it will be convenient to call the "southern" one) iff the dual arrangement of hemispheres u_i^* is *feasible*, i.e., if the intersection of the hemispheres is nonempty: A vector ν (which we think of as the north pole) lies in (the interior of) that intersection iff U is contained in (the interior of) the (southern) hemisphere $\nu^* = \{x \in \mathbf{S}^d : \langle \nu, x\rangle \leq 0\}$.

Combining these correspondences, we have the two duality transforms. The first one between points in $\mathbf{R}^d$ (the "southern" copy of $\mathbf{R}^d$) and feasible arrangements of hemispheres in $\mathbf{S}^d$. The second one between the two copies of $\mathbf{R}^d$ tangent to $\mathbf{S}^d$ at the "north pole" and the "south pole", respectively (the primal and the dual plane).

2.1.2. *"Vertical" point-hyperplane duality.* A special case of this duality arises if we chose the origin in $\mathbf{R}^d$ to lie at $x_d = -\infty$. Then a set S of n points in $\mathbf{R}^d$ corresponds to n non-vertical affine hyperplanes, where we implicitly associate the lower halfspace with each hyperplane. Thus, a point is at level k in the arrangement if there are exactly k hyperplanes that pass below it, where "vertical", "upper", "lower", "above", and "below" refer to the x_d-coordinate. In coordinates, a point

[6] If all $u \in U$ are not contained in one hemisphere, we can still apply a radial projection, but we get a "signed point configuration" S in $\mathbf{R}^d$, where each point $p \in S$ corresponding to a vector $u \in U$ carries a sign ± 1, depending on whether u is contained in the hemisphere or not.

$S=\{p_1,\dots,p_n\}$ (n points in $\mathbf{R}^d$), plus origin o	$\leftrightarrow$	$\mathcal{A}=\{H_1,\dots,H_n\}$ (n halfspaces in $\mathbf{R}^d$), with origin $o\in\bigcap_{i=1}^n H_i$
halfspace H with $o\notin H$, $B=\{i:p_i\in\partial H\}$, $C=\{i:p_i\notin H\}$	$\leftrightarrow$	point $p\in\mathbf{R}^d$, $B=\{i:p\in\partial H_i\}$, $C=\{i:p\notin H_i\}$
k-facet σ of S with $o\in\sigma^-$ ($\lvert S\cap\sigma^+\rvert=k$)	$\leftrightarrow$	vertex v of $\mathcal{A}$ at level k

Table 2. Dictionary for polar point-halfspace duality

$(a_1,\dots,a_d)$ corresponds to the hyperplane $\{x\in\mathbf{R}^d : x_d=a_1x_1+\dots,a_{d-1}x_{d-1}\}$ (or, more precisely, to the lower halfspace $\{x\in\mathbf{R}^d : x_d\leq a_1x_1+\dots,a_{d-1}x_{d-1}\}$), and vice versa. A point p lies above/on/below a non-vertical hyperplane h iff the dual point h^* lies above/on/below the dual hyperplane p^*. In particular, a k-facet of S corresponds to a vertex of the arrangement at level k if it is an "lower k-facet", i.e., if the k points lie below it; otherwise, it corresponds to a vertex at level $n-d-k$.

2.2. The k-Set Polytope. For $S\subset\mathbf{R}^d$, define

$$P_k(S):=\operatorname{conv}\left\{\sum X : X\subseteq S, |X|=k\right\},$$

where $\sum X$ is a shorthand for $\sum_{x\in X}x$. If S is finite, then $P_k(S)$ is a convex polytope, which is known as the *k-set polytope* because of Proposition 2.1 below, and it is for this finite case that $P_k(S)$ was first defined by Edelsbrunner, Valtr, and Welzl [**59**], who used it to obtain improved bounds for the number of halving facets of so-called *dense* point sets in $\mathbf{R}^d$, $d\geq 3$. However, $P_k(S)$ is also of interest in the infinite case. For instance, Onn and Sturmfels [**112**] studied the set $P_k(\mathbf{N}_0^d)$, which is no longer a bounded polytope but still a convex polyhedron, in relation with computational commutative algebra.

Note that for $0<k<|S|$, $\dim(P_k(S))=\dim(\operatorname{aff}(S))$. It is not difficult to prove the following characterization of the vertices and facets of $P_k(S)$ (cf. [**21**]), and see Figure 13 for an illustration).

Proposition 2.1. (1) *A point $v\in\mathbf{R}^d$ is a vertex of $P_k(S)$ iff $v=\sum A$ for a k-set A of S.*

(2) *A facet F of $P_k(S)$ corresponds to a closed halfspace H, spanned by points from S, such that $|\operatorname{int}H\cap S|=j<k$ and $|H\cap S|>k$ (where $\operatorname{int}H$ denotes the interior of H, i.e., the corresponding open halfspace). More precisely,*

$$F=P_{k-j}(\partial H\cap S)+\sum(\operatorname{int}H\cap S).$$

Every facet of $P_k(S)$ is of this form, and conversely, each H as above gives rise to a facet.

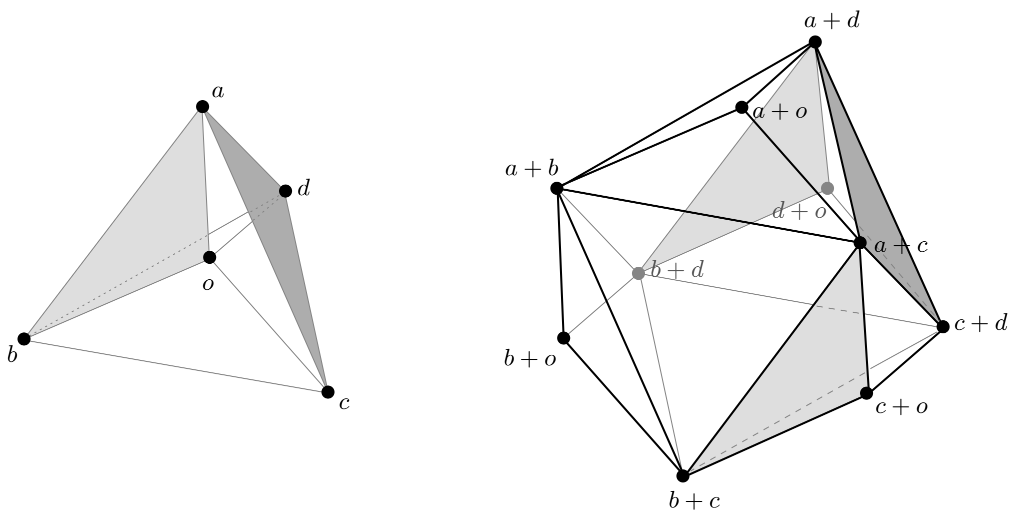


FIGURE 13. A set of five points in $\mathbf{R}^3$ (the vertices of a regular tetrahedron and its center) and the corresponding 2-set polytope P_2. The 0-facet *acd* corresponds to the facet $\operatorname{conv}\{a+c, a+d, c+d\}$ of P_2; the halving facet *abo*, depending on its coorientation, corresponds to two antipodal facets $\operatorname{conv}\{a+c, b+c, c+o\}$ and $\operatorname{conv}\{a+d, b+d, d+o\}$ of P_2.

2.2.1. *(i,j)-Partitions.* Thus, if S is a finite set in general position, then each facet of $P_k(S)$ corresponds to a j-facet of S with $k-d < j < k$. The faces of intermediate dimension can be characterized in terms of so-called *(i,j)-partitions* (see Andrzejak and Welzl [**21**]): These are pairs (I, J) of subsets of S such that $|I| = i$, $|J| = j$, and there is an affine halfspace H such that I consists of the points of S on the boundary of H and J of the points of S in the interior of H. Thus, in the dual arrangement of hemispheres, such a partition corresponds to a $(d-i)$-dimensional face at level j. In particular, the (d, j)-partitions are precisely the j-facets and the $(0, j)$-partitions are the j-sets of S.

2.2.2. *Linear identities.* The Euler-Poincaré formula gives a linear relation between the face numbers of a general convex polytope P, $\sum_{r=-1}^{d} f^r(P) = 0$. Based on this, Andrzejak and Welzl derive a number of linear relations for the numbers of (i,j)-partitions, $0 \le i \le d$, $0 \le j \le n-i$. In particular, in dimensions 2 and 3, the k-set polytope $P_k(S)$ is simplicial, which implies that the numbers of vertices and of facets of $P_k(S)$, respectively, determine each other: In dimension 2, this yields another proof of $a_k = e_{k-1}$, $1 \le k \le n-1$, and in dimension 3, one obtains the linear relation $a_k = \frac{1}{2}(e_{k-2} + e_{k-1})$ for $1 \le k \le n-1$. Andrzejak and Welzl also show that in higher dimensions, the numbers of k-facets and of k-sets no longer determine each other.

Mulmuley [**107**] obtained a set of linear relations for the numbers of i-dimensional faces at level j in an affine arrangement of halfspaces, $0 \le r \le d$ and $0 \le j \le k$, under the assumption that all these faces are bounded. We will return to this in Section 10.

Another linear relation of Euler-Poincaré type was obtained by Gulliksen and Hole [**73**], who showed that for every finite set of points in general position in *odd dimension* d, $\sum_k (-1)^k a_k = 0$, where a_k denotes the number of k-sets.

2.3. Depth, Center Points, and Tverberg's Theorem. Fix a set S of n points in $\mathbf{R}^d$. For $x \in R^d$, the (*Tukey* or *halfspace*) *depth* of x with respect to S is defined as $\min_{x\in H} |S \cap H|$, where H ranges over all halfspaces that contain x. Let $C_r(S)$ denote the set of all points in $\mathbf{R}^d$ of depth at least r with respect to S. (For instance, $C_1(S) = \mathrm{conv}(S)$.) Then $C_r(S) = \bigcap \mathcal{H}_{n-r+1}$, where $\mathcal{H}_j$, $0 \leq j \leq n$ denotes the (infinite) set of closed halfspaces that contain exactly j points of S.

2.3.1. *Centerpoints.* If $j > \frac{dn}{d+1}$, or equivalently $r = n - j + 1 \leq \lceil n/(d+1) \rceil$, then any $d+1$ halfspaces in $\mathcal{H}_j$ have a point of S in common and in particular a nonempty common intersection, so by Helly's Theorem[7], $C_r(S) = \bigcap \mathcal{H}_j \neq \emptyset$. The set $C_{\lceil \frac{n}{d+1} \rceil}(S)$ is called the *center* of S, and its elements are called *centerpoints*.[8] They can be seen as the higher-dimensional generalization of the *median* of n real numbers (the case $d = 1$).

Being the intersection of closed halfspaces, $C_r(S)$ is a convex set. In fact, by the following lemma it is a convex polyhedron (see Agarwal, Sharir, and Welzl [**6**]).

LEMMA 2.2. *Let S be a set of n points in general position in $\mathbf{R}^d$.*

(1) *Let $\overline{\mathcal{H}}_j \subset \mathcal{H}_j$ be the set of all closed halfspaces whose bounding hyperpane is spanned by S and that contain exactly j points of S. Then*
$$C_r(S) = \bigcap \overline{\mathcal{H}}_{n-d-r+1} = \bigcap_{(r-1)\text{-facets } \sigma} \overline{\sigma^-}.$$

(2) *Each facet of the polytope $C_r(S)$ is contained in a unique $(r-1)$-facet σ of S. Moreover, out of the potentially many $(r-1)$-facets incident to a given $(d-2)$-dimension simplex ρ spanned by S, at most two can determine facets of the polytope $C_r(S)$. Therefore, the number of facets of the polytope is at most $\frac{2}{d+1}\binom{n}{d-1}$.*

2.3.2. *Tverberg's Theorem.* Radon's Lemma states that any set of $n \geq d+2$ points in $\mathbf{R}^d$ can be partitioned into two subsets whose convex hulls intersect. Tverberg proved a far-reaching generalization of this result. A partition $S = S_1 \cup \ldots \cup S_r$ of S into r disjoint subsets is called a *Tverberg partition* if $\bigcap_{i=1}^r \mathrm{conv}(S_i) \neq \emptyset$. In this case, any point in the intersection is called an (r-)*Tverberg point.* Each halfspace containing an r-Tverberg point x constains at least one point from each $S-i$, so x has depth at least r in S.

THEOREM 2.3 (Tverberg's Theorem). *Any set of $n \geq (d+1)(r-1)+1$ points in $\mathbf{R}^d$ has a Tverberg partition into r parts.*

Note that Radon's Lemma is the special case $r = 2$ of Tverberg's Theorem. For $r > 2$, the partition is generally not unique. Observe that setting $r = \lceil \frac{n}{d+1} \rceil$, Tverberg's Theorem also implies the existence of centerpoints. Tverberg's original

[7] Actually, a little care is needed, since there are infinitely many halfspaces and in general, the infinite version of Helly's Theorem is false for noncompact sets. However, each halfspace $h \in \mathcal{H}_j$ can be replaced by the compact set $\mathrm{conv}(h \cap S)$, see [**99**]; alternatively, it suffices to consider a finite subset of $\mathcal{H}_j$, see below.

[8] The argument is tight, as we see by placing a tiny cloud of $n/(d+1)$ points at each of the vertices of a d-dimensional simplex.

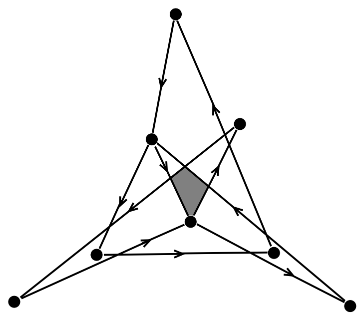


FIGURE 14. A set of eight points in the plane, the 2-edges, and the center C_3.

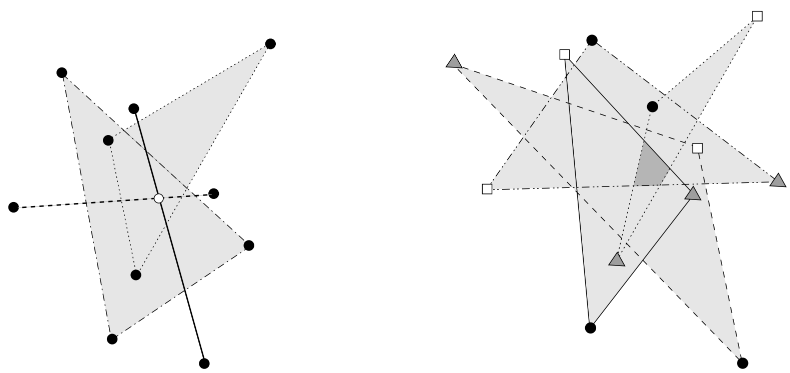


FIGURE 15. On the left: A set of ten points in the plane and a Tverberg partition into four parts. On the right: A set $S = C_1 \cup C_2 \cup C_3$ of three times four points in the plane and a colorful Tverberg partition into four rainbow parts (the points in each class C_i are drawn as small black dots, grey triangles, and white squares, respectively).

proof was a complicated continuous motion argument, but subsequently, several simpler proofs were found (see, for instance, Kalai [**83**] for a survey).

Motivated by the halving facet problem in higher dimensions, Bárány, Füredi, and Lovász [**28**] suggested the following "colorful" version of Tverberg's result:

THEOREM 2.4 (Colorful Tverberg Theorem). *For all positive integers r, d, there exists a number $t = t(r, d)$ (the* colorful Tverberg number*) such that the following holds: Let $C_1, \ldots, C_{d+1}$ be disjoint sets of of t points each in $\mathbf{R}^d$ (which we think of as "color classes"), and let $S = C_1 \cup \ldots \cup C_{d+1}$. Then there are r disjoint "rainbow subsets" $S_1, \ldots, S_r$ of S whose convex hulls intersect, $\bigcap_{i=1}^{r} \operatorname{conv}(S_i) \neq \emptyset$, where "rainbow" means that $|S_i \cap C_j| = 1$ for all $1 \leq i \leq r$ and $1 \leq j \leq d + 1$.*

Bárány et al. proved the case $d = 2$ and used it to prove the first nontrivial upper bound for the number of halving facets in three dimensions. The general Colorful Tverberg Theorem was established by Živaljević and Vrećica [**146**]. Bárány et al. conjectured that $t(r, d) = r$ always suffices. This was proven by Bárány and

Larman for the case $d = 2$, and by Lovász for $r = 2$, see [**27**]. The approach of Živaljević and Vrećica yields $t(r, d) \leq 2r - 1$ if r is a prime (this has been extended to prime powers), which implies $t(r, d) \leq 4r - 1$ for general r (by "Bertrand's Postulate", there is always a prime between r and $2r$). The reason for the sudden and surprising appearance of primality assumptions is that the proofs use methods from equivariant topology.

2.3.3. *A glimpse of topological arguments.* Let $K^{(d+1)}(n)$ be the family of all subsets of $[n]$ of size $d+1$, and let $K^{(d+1)}(t, \ldots, t)$ the collection of transversals of a ground set of $(d+1)t$ elements partitioned into $(d+1)$ classes of t elements each (a transversal is a set that contains exactly one element from each class). For a combinatorialist, these are the *complete* $(d+1)$*-uniform hypergraph on* n *elements* and the *complete* $(d+1)$*-partite* $(d+1)$*-uniform hypergraph on* $t + \ldots + t$ *elements*, respectively. From a topological point of view, such a hypergraph K can be seen as simplical complex K (by including, for each set in K, all it subsets) and hence as a topological space.

From this point of view, Tverberg's Theorem and its colorful variant are statements of the following form: If a complex K satisfies certain assumptions then for every affine map $f : \mathsf{K} \to \mathbf{R}^d$ (induced by a the placement of the vertices), there are r vertex-disjoint faces of K whose images under the map f have a common point of intersection. It is natural to ask if the conclusion that there is such an *r-fold point* still holds true if instead of affine maps, one considers general continuous maps. For $\mathsf{K}^{(d+1)}(n)$ and $= \mathsf{K}^{(d+1)}(t, \ldots, t)$, this is the content of the Topological Tverberg Theorem and the Topological Colored Tverberg Theorem, respectively.

Very roughly speaking, the strategy for proving the existence of such r-fold points of intersection is this: With the complex K, one associates a new complex $\mathsf{K}^r_{\mathrm{del}}$ and with $\mathbf{R}^d$ a new space $(\mathbf{R}^d)^r_{\mathrm{del}}$ such that the cyclic group $\mathbf{Z}_r$ on r elements acts on both of these spaces. One way of doing this is by deleted products: Then the faces of $\mathsf{K}^r_{\mathrm{del}}$ are products $F_1 \times \ldots \times F_r$ of r pairwise disjoint faces F_i of K, and similarly, the space $(\mathbf{R}^d)^r_{\mathrm{del}}$ consists of all r-tuples of pairwise distinct points in $\mathbf{R}^d$. The group acts by interchanging components. Primality of r plays a role is that it makes the action satisfy the technical condition of being fixpoint-free. Now, these two new spaces have the property that if there was a "bad" continuous map $f : \mathsf{K} \to \mathbf{R}^d$ *without any r-fold point*, then it would induce (by applying f componentwise) an *equivariant map* $\tilde{f} : \mathsf{K}^r_{\mathrm{del}} \to (\mathbf{R}^d)^r_{\mathrm{del}}$,i.e., a continuous map compatible with the group action on both spaces.

In order to arrive at a contradiction (and thus to show that r-fold points always exist), one argues that under the assumptions made on K, the space $\mathsf{K}^r_{\mathrm{del}}$ (together with the group action) is "complicated" while the space $(\mathbf{R}^d)^r_{\mathrm{del}}$ is "simple", and that the map $\tilde{f}$ preserves too much structure to map a complicated space into a simple one.

For the purposes of illustration, consider the case $r = 2$ of the Topological Tverberg Theorem (i.e., the Topological Radon Lemma): Then $(d+1)(r-1)+1 = d+2$, and $\mathsf{K}^{d+1}(d+2)$ consists of all $d+1$-element subsets of a ground set of a ground set of $d + 2$ elements. Thus, if we consider it as a simplicial complex by adding all the smaller subsets, it is the d-dimensional boundary complex $\partial\Delta$ of a $(d+1)$-dimensional simplex Δ. We want to show that for any continuous map $\partial\Delta \to \mathbf{R}^d$, there are two disjoint faces of $\partial\Delta$ whose images intersect. We can represent $(\partial\Delta)^2_{\mathrm{del}}$ geometrically as the boundary of the Minkowski sum $\Delta + (-\Delta)$ (if F_1 and F_2 are disjoint faces of the simplex Δ, then there is a linear function on Δ that attains its maximum on F_1 and its minimum on F_2), so the deleted product of $\mathsf{K}^{d+1}(d+2)$ is homeomorphic to the d-dimensional sphere $\mathbf{S}^d$, and the $\mathbf{Z}_2$ action $F_1 \times F_2 \leftrightarrow F_2 \times F_1$ of corresponds to the antipodal map on $\mathbf{S}^d$. On the other hand, the deleted product $(\mathbf{R}^d)^2_{\mathrm{del}}$ is "simple" because $(x, y) \mapsto \frac{x-y}{\|x-y\|}$ defines a $\mathbf{Z}_2$-equivariant map into the $(d-1)$-dimensional sphere $\mathbf{S}^{d-1}$, again with the antipodal action (in fact, this yields a $\mathbf{Z}_2$-homotopy equivalence). Thus, a "bad" map f without a double-point would induce a $\mathbf{Z}_2$-equivariant map $\mathbf{S}^d \to \mathbf{S}^{d-1}$. Such a map cannot exist, by the classical Borsuk-Ulam Theorem.

Needless to say, our extremely vague general outline barely scratches the surface of the subject. At the very best, we hope that it conveys some of the general flavor of the method. For a serious (and very readable) introduction to topological methdos in combinatorics, we refer the reader to Matoušek's book [**100**] (from which we also took the example of the Topological Radon Lemma) and to Živaljević's survey articles [**139, 140**].

2.3.4. *Algorithmic aspects.* Teng [**136**] showed that if d and r are considered as part of the input, then given S and x in $\mathbf{R}^d$, it is coNP-complete to determine whether x is a centerpoint of S, and NP-complete to decide whether x is an r-fold Tverberg point of S. Clarkson, Eppstein, Miller, Sturtivant, and Teng [**49**] gave a polynomial-time algorithm for computing an approximate centerpoint in any dimension. As for computing a single centerpoint in fixed dimension, Jadhav and Mukhopadhyay [**79**] gave a linear algorithm for the planar case; Naor and Sharir [**108**] describe an $O(n^2 \operatorname{polylog}(n))$-algorithm for three dimensions; and Chan [**41**] presented a randomized algorithm that, in any fixed dimension d, computes a point of maximal depth in time $O(n \log n + n^{d-1})$.

Matoušek [**97**] gave an $O(n(\log n)^4)$-algorithm to compute the whole center $C_{\lceil n/3 \rceil}$ of a planar point set, and Agarwal, Sharir, and Welzl [**6**] developped an algorithm that solves this task in three dimensions in time $O(n^{2+\varepsilon})$, for any fixed $\varepsilon > 0$.

In any fixed dimension d, Tverberg's existence proof can be turned into an $n^{O(d^2)}$ algorithm for computing a Tverberg point. Agarwal et al. presented a algorithm that, given $3n$ points in a plane, partitioned into three classes of n points each, determines in time $O(n^{11})$ if a given point is a colored Tverberg point.

2.4. Gale Duality.

Observation 2.5 (Gale Duality). *Let $V = \{v_1, \ldots, v_n\}$ be a configuration of n vectors in $\mathbf{R}^r$. We do not assume that the vectors are in general position, or even pairwise distinct, only that V is full-dimensional, i.e., that the v_i's linearly span $\mathbf{R}^r$. Then there is a (full-dimensional) configuration $W = \{w_1, \ldots, w_n\}$ of n vectors in $\mathbf{R}^{n-r}$ with the following property: The space of* linear dependencies *of V equals the space of linear valuations of W, i.e., for all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{R}^n$,*

$$\sum_i \lambda_i v_i = 0 \iff \exists u \in \mathbf{R}^{n-r} : \lambda_i = \langle u, w_i \rangle, 1 \le i \le n,$$

and vice versa. The vector configurations V and W determine one another up to bijective linear transformations of $\mathbf{R}^r$ and $\mathbf{R}^{n-r}$, respectively, and are called Gale duals *of each other.*

To see this, interpret V as an $(r \times n)$-matrix with columns v_i. The rows of V span an r-dimensional subspace of $\mathbf{R}^n$. Pick a basis for the orthogonal complement of this row space; this yields $n-r$ basis vectors of length n. Interpret these as rows of an $((n-r) \times n)$-matrix W and read out the w_i's as the columns of this matrix.

Note that V is in general position (i.e., any r or fewer of the v_i's are linearly independent) iff its Gale dual W is. We remark that unlike in the case of polar duality, in Gale duality each dual vector w_i depends on the entire primal configuration V.

2.4.1. *Perfectly centered configurations.* The following is general-position analogue of centrally symmetric point sets. Let $o \in \mathbf{R}^d \setminus S$ be such that $S \cup \{o\}$ is in general position. We say that o is a *perfect centerpoint* of S (or that S is *perfectly centered* with respect to o)[9] if the depth of o with respect to S is $\lfloor \frac{n-d+1}{2} \rfloor$.

Observation 2.6. *The following statements are equivalent:*

(1) *o is a perfect centerpoint of S.*
(2) *All j-facets of $S \cup \{o\}$ incident to o are* halving *in the sense that $j \in \{\lfloor \frac{n-(d-1)}{2} \rfloor, \lceil \frac{n-(d-1)}{2} \rceil\}$.*
(3) *$o \in \mathrm{conv}(R)$ for every $R \subseteq S$ with $|R| \geq \lceil \frac{n+d+1}{2} \rceil$.*
(4) *The Gale dual of S/o is neighborly, where S/o is the d-dimensional configuration of vectors $p-o$, $p \in S$. In particular, since there are neighborly polytopes of any dimension and with any number of vertices, perfectly centered point configurations exist for all $n \geq d+1$.*

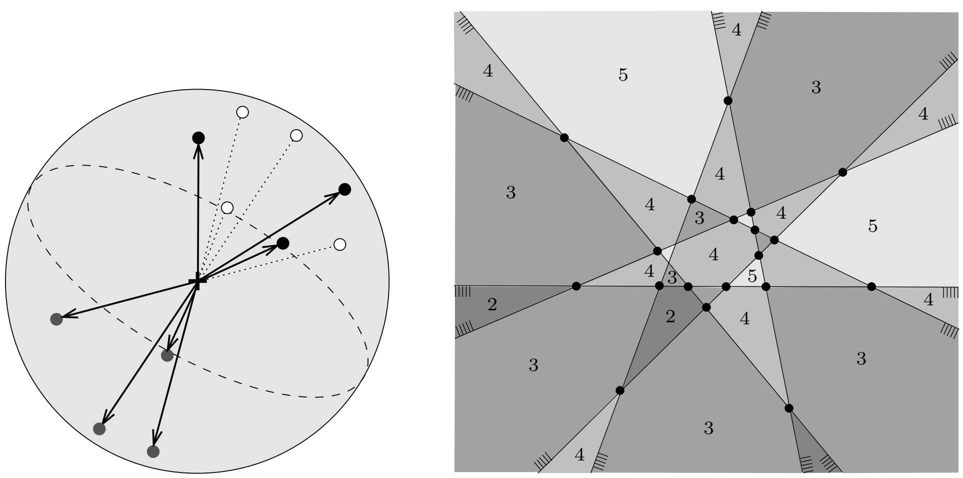


Figure 16. A perfectly centered vector configuration S/o in $\mathbf{R}^3$ (which can be seen as a signed point configuration in two dimensions) and the polar dual arrangement.

[9]This depth is best possible if we want general position: $C_{\lfloor \frac{n-d+2}{2} \rfloor}$ might be nonempty, for instance if the case of a regular n-gon in the plane, n even, but the region will not have a nonempty interior.

COROLLARY 2.7. *Let S be a set of n points in $\mathbf{R}^d$ with perfect centerpoint o. Let $\mathcal{A}$ be the $(d-1)$-dimensional arrangement of hemispheres $\{x \in \mathbf{S}^{d-1} : \langle x, p-o \rangle = 0\}$ with outer normal vectors $p-o$, $p \in S$, see Figure 16. Then all vertices of $\mathcal{A}$ lie at the middle levels $\lfloor \frac{n-(d-1)}{2} \rfloor, \lceil \frac{n-(d-1)}{2} \rceil$.*

3. Random Sampling

Beginning with the seminal papers by Clarkson [**46**], Hausler and Welzl [**77**], and Clarkson and Shor [**50**], random sampling has proven to be a versatile and powerful tool in discrete and computational geometry. For a rather general unified framework for a variety of random sampling arguments, see Sharir [**123**]. Here, we just restrict ourselves to two types of random-sampling results of particular relevance to k-sets.

3.1. Crossing-Lemma-Type Results. One group of applications of random sampling comprises the Crossing Lemma and its generalizations.

LEMMA 3.1 (Crossing Lemma). *If $G = (S, E)$ is a simple graph on n vertices, then either $|E| = O(n)$, or in any drawing of G in the plane, there are at least $\Omega(|E|^3/n^2)$ crossings.*

The Crossing Lemma was originally conjectured by Erdős and Guy [**65**] and proved by Ajtai, Chvatal, Newborn, and Szemeredi [**14**] and, independently, by Leighton [**93**]. Székely [**133**] was among the first to demonstrate its power and usefulness by giving strikingly simple proofs for a number of difficult results on incidence and distance problems. In the context of this survey, the Crossing Lemma plays an essential role in the proof of Dey's upper bound on the maximum number of k-sets in the plane as well as in the subsequent improved bounds for the maximum number of k-sets in 3 and 4 dimensions, all of which we will discuss below.

The following probabilistic proof of the Crossing Lemma is due to Chazelle, Sharir, and Welzl, see Aigner and Ziegler [**13**]. Let G be a simple graph with vertex set S and edge set $E \subseteq \binom{S}{2}$. Consider a fixed drawing of G in the plane, and let X be the set of crossings in the drawing; without going into the subtleties of the precise definition of a drawing of a graph, the properties that we need are the following: any two edges cross at most once, and edges with a common endpoint do not cross (we may assume this without loss of generality if we consider drawings with the minimal number of crossings). Because of these assumptions and because the graph is simple, a crossing can be identified with the set of four endpoints of the crossing edges, i.e., $X \subseteq \binom{S}{4}$.

For every $R \subseteq S$, the given drawing of G induces a drawing of the induced subgraph $G[R]$ that is induced by the given drawing of G. By Euler's formula, a simple planar graph on r vertices has at most $3r$ edges. Consequently,

$$|X[R]| \geq |E[R]| - 3|R|$$

for all subsets $R \subseteq S$. Here we use the notation $A[R] := A \cap \binom{S}{d}$ for the restriction of a collection $A \subseteq \binom{S}{d}$ of d-tuples in S to a subset $R \subseteq S$. Note that we are crucially using a kind of "locality property" here: a crossing in the induced drawing of the subgraph $G[R]$ corresponds to an original drawing of G all of whose endpoints belong to R; thus, the (combinatorially defined) set $X[R]$ indeed equals the (geometrically defined) set of crossings in the induced drawing.

Now let $0 < p \leq 1$ and let R_p be a random subset of S, where each element is chosen independently with probability p. Then by linearity of expectation, the above inequality implies

$$\underbrace{\mathbf{E}[|X[R_p]|]}_{p^4|X|} \geq \underbrace{\mathbf{E}[|E[R_p]|]}_{p^2|E|} - 3\underbrace{\mathbf{E}[|R_p|]}_{pn}$$

Now either $|E| < 4n$, say, or we can choose $p = 4n/|E| \leq 1$, and solve the last inequality to $|X| \geq \frac{1}{64}\frac{|E|^3}{n^2}$, as desired.

More generally, suppose that we are given a ground set S of n elements, a set T of "configurations" (taking the place of edges in a graph), and a set X of "conflicts" or "crossings". We assume that the configurations and the crossings are defined by fixed numbers of elements from the ground set, i.e., we can view them as collections $T \subseteq \binom{S}{b}$ and $X \subseteq \binom{S}{c}$ of b-element and c-element subsets of S, respectively, for some fixed integers b, c. In this setting, virtually the same proof as before yields the following:

Lemma 3.2 (Abstract Crossing Lemma). *Let $|S| = n$, $T \subseteq \binom{S}{b}$ and $X \subseteq \binom{S}{c}$. Assume that for some integer a, a linear inequality*

$$|X[R]| \geq \lambda \cdot |T[R]| - \mu\binom{|R|}{a}$$

with coefficients $\mu, \lambda > 0$ holds for all subsets $R \subseteq S$. Then either $|T| = O(n^a)$ or

$$X \geq \Omega\big(n^{a\frac{b-c}{b-a}}|T|^{\frac{c-a}{b-a}}\big).$$

For comparison, note that in the classical Crossing Lemma, we have $a = 1$, $b = 2$ and $c = 4$. In our applications, the crossings X will be geometrically defined, and we will need the same kind of locality property as before, namely that in the restriction to any subset $R \subseteq S$, the geometrically defined conflicts coincide with the combinatorially defined crossings $X[R]$.

3.2. The Clarkson-Shor Method. This method was introduced in [**47, 50**]. The abstract setting is the following. Let S be a set of n elements, and let $T \subseteq S^d$ be a set of "configurations", each defined by d elements of S for some integer d. Suppose furthermore that we have a "conflict relation" between elements $a \in S$ and configurations $t \in T$, formally a subset $X \subseteq S \times T \subseteq S^{d+1}$, where we assume that none of the d elements defining a configuration is in conflict with it.

The *weight* of a configuration is defined as the number of elements in conflict with it, and we write $T_k = T_k(S)$ and $T_{\leq k} = T_{\leq k}(S)$ for the set of configurations with weight exactly k and at most k, respectively.

For a subset $R \subseteq S$, we define $T_k(R)$ to consist of those configurations in $T[R] := T \cap R^d$ that have weight k with respect to the conflict relation $X[R] := X \cap R^{d+1}$. Thus, a configuration belongs to $T_k(R)$ if the defining points belong to R and it is in conflict with exactly k points in R. (We do not care if the configuration is also in conflict with some other points that do not belong to R.)

Proposition 3.3. *Suppose that $|T_0(R)| \leq C \cdot \binom{|R|}{c}$ for all $R \subseteq S$, for some constant $C > 0$ and some positive integer $c \leq d$. Then*

$$|T_{\leq k}(S)| \leq C \cdot \big(\frac{e}{d-c}\big)^{d-c}\binom{n}{c}(k+d-c)^{d-c},$$

where e is the basis of the natural logarithm. In particular, if d and c are fixed and $n \to \infty$, then $|T_{\leq k}(S)| = O(n^c(k+1)^{d-c})$.

PROOF. Pick a random subset $R_p \subseteq S$ by including each element independently with probability p, for some probability $0 < p < 1$. Then $\mathbf{E}[|T_0(R)|] \leq Cp^c\binom{n}{c}$, by assumption and the remark following Lemma **??**. On the other hand, a configuration $\tau \in T_j(S)$ belongs to $T_0(R_p)$ with probability $p^d(1-p)^j$ (we have to include the d elements defining t and to exclude the j elements with which τ is in conflict. Thus, by linearity of expectation,

$$\mathbf{E}[|T_0(R)|] = \sum_j p^d(1-p)^j|T_j(S)| \geq p^d(1-p)^k \sum_{j \leq k}|T_j(S)| = p^d(1-p)^k|T_{\leq k}(S)|.$$

Combining this with the preceding upper bound for $\mathbf{E}[|T_0(R_p)|]$, we obtain $|T_{\leq k}(S)| \leq Cp^{-d}(1-p)^{-k}\binom{n}{c}$, and substituting $p = (d-c)/(k+d-c)$ yields the claimed bound. $\square$

For example, S might be a set of n points in general position in $\mathbf{R}^d$, the configurations in T could be the $(d-1)$-dimensional cooriented simplices (each of them spanned by a d-tuple of points of S, with the coorientation determined by an ordering of the defining points, say), and $p \in S$ and $\sigma \in T$ are in conflict iff $p \in \sigma^+$. In this case, T_k is precisely the set of k-facets of S.

By the Upper Bound Theorem for convex polytopes, $e_0(R) \leq 2\binom{|R|}{\lfloor d/2 \rfloor}$ for any $R \subseteq S$. Therefore:

COROLLARY 3.4. *For any set S of n points in general position in $\mathbf{R}^d$,*

$$e_{\leq k}(S) \leq 2\big(\frac{e}{\lceil d/2 \rceil}\big)^{\lceil d/2 \rceil}\binom{n}{\lfloor d/2 \rfloor}(k + \lceil d/2 \rceil)^{\lceil d/2 \rceil}.$$

For fixed dimension d, this upper bound is tight up to constant factors, as shown by any neighborly set of points, see Proposition 4.1.

REMARK 3.5 (($\leq k$)-Levels for curves.). The same kind of argument also implies upper bounds for the complexity of the ($\leq k$)-level in arrangements of other curves and surfaces, whenever good bounds for the complexity of the 0-level are available. For instance, for an arrangement of n pseudocircles in the plane, the ($\leq k$)-level has complexity at most $O(nk)$, because the zero-level in such arrangements (i.e., the boundary of the union of the enclosed disks) has at most linear complexity; similarly, for any constant s, the ($\leq k$)-level in an arrangement of n x-monotone ($\leq s$)-intersecting curves has complexity $O(k^2\lambda_s(n/k))$, where $\lambda_s(m)$ is a function that is almost linear in m for any fixed s, because the 0-level (or *lower envelope*) in such arrangements has complexity at most $O(\lambda_s(n))$, as shown by Agarwal, Sharir, and Shor [**5**]; for the details, see Sharir [**122**].

Upper Bounds for $e_{1/2}$ imply upper bounds for e_k. It is easy to see that for all $e_k^{(d)}(n) \leq 2e_{1/2}^{(d)}(2n-d)$.[10] Using a Clarkson-Shor-type random sampling argument, Agarwal, Aronov, Chan and Sharir [**2**] showed how upper bounds for the $e_{1/2}$ imply upper bounds for e_k that are sensitive to k:

[10] We may assume that $k \leq (n-d)/2$. Given a set S of n points in $\mathbf{R}^d$, choose a point $o \notin S$ such that at least half of the k-facets of S contain o on their positive side. Let S' be a set of $n-d-2k$ new points very close to o, and set $S'' := S \cup S'$. Then every k-facet of S with o on its positive side becomes a halving facet of S''.

THEOREM 3.6. *Let the dimension d be fixed. If $e^{(d)}_{1/2}(n) = O(n^{d-c_d})$ for some constant c_d, then*

$$e^{(d)}_k(n) = O(n^{\lfloor d/2\rfloor}(k+1)^{\lceil d/2\rceil - c_d})$$

for all k.

4. Special Point Sets

4.1. Neighborly Point Sets. For any set of n points in convex position in the plane (i.e., the vertex set of a convex n-gon), $e_k = n$ for $0 \le k \le n-2$. If S is a set of n points in general and convex position in $\mathbf{R}^3$, i.e., in general position and the vertex set of a convex 3-polytope, then $e_k(S) = 2(k+1)n - 4\binom{k+2}{2}$; this is maybe less obvious than the planar statement (for one thing, there are many different combinatorial types of simplicial 3-polytopes on n vertices). In higher dimensions, though, convex position is no longer sufficient to determine e_k as a function of n, k, and d; the right generalization is neighborliness.

A set S of n points in $\mathbf{R}^d$ is called *neighborly* if every subset $F \subseteq S$ of cardinality $|F| \le \lfloor d/2 \rfloor$ determines a face of the convex hull of S, i.e., there is a closed halfspace H such that $S \subset H$ and $S \cap \partial H = F$ (it follows from Radon's Theorem that $\lfloor d/2 \rfloor$ is the largest integer for which this definition makes sense).

PROPOSITION 4.1. *Let S be a neighborly set of n points in general position in $\mathbf{R}^d$. Then $e_k(S)$ depends only on n, k, and d. Specifically,*

$$e_k(S) = \begin{cases} 2\binom{k+\lceil d/2\rceil-1}{\lceil d/2\rceil-1}\binom{n-k-\lceil d/2\rceil}{\lceil d/2\rceil-1}, & \textit{for odd } d,\\[2ex] \binom{k+d/2-1}{d/2-1}\binom{n-k-d/2}{d/2} + \binom{k+d/2}{d/2}\binom{n-k-d/2-1}{d/2-1}, & \textit{for even } d. \end{cases}$$

This follows from a Clarkson-Shor-type random sampling argument. First, let S be an arbitrary of n points in $\mathbf{R}^d$, and consider a subset R chosen uniformly at random from $\binom{S}{r}$. For any fixed k-facet σ of S, the probability that σ is a 0-facet of R equals $\binom{n-d-k}{r-d}/\binom{n}{r}$. Thus, by linearity of expectation, $\mathbf{E}_{R\in\binom{S}{r}}[e_0(R)] = \sum_k \binom{n-d-k}{r-d}/\binom{n}{r} e_k(S)$, $r = d+1, \ldots, n$. This is an invertible system of linear equations relating the vector $(e_k(S) : 0 \le k \le n-d)$ and the vector $(\mathbf{E}_{R\in\binom{S}{r}}[e_0(R)] : d \le r \le n)$.

The second observation is that neighborliness is inherited by subsets. Thus, by (the characterization of the extreme cases in) the Upper Bound Theorem for convex polytopes, if S is neighborly then the number of convex hull facets of any r-element subset is the same number depending only on r and d, namely $e_0(R) = \binom{r-\lceil d/2\rceil}{\lfloor d/2\rfloor} + \binom{r-\lceil (d+1)/2\rceil}{\lfloor (d-1)/2\rfloor}$. To compute the explicit numbers $e_k(S)$ for a neighborly point set, one can either solve the above system of linear equations or count the k-facets for a particular family of neighborly point sets (e.g., points on the moment curve, see Andrzejak and Welzl [**21**]).

4.2. Random Points Sets. Bárány and Steiger [**29**] studied the expected number $e_k(\mu, n)$ of k-facets for a set of n independent random points drawn from a probability distribution μ in $\mathbf{R}^d$ (the assumption of μ corresponding to general position is that every hyperplane has measure zero).

Bárány and Steiger show that $e_k(\mu, n) = O(n^{d-1})$ if μ is spherically symmetric. In the plane, they also prove that if μ_K is the uniform distribution on a convex body K in the $\mathbf{R}^2$, then $e_k(\mu_K, n) = \Theta(n)$ if k is linear in n; at the other end of the range, they show that $e_k(\mu_K, n) = \Theta(e_0(\mu_K, n))$ for $k = O(1)$.

On the other hand, if the probability distribution can be arbitrary, then there is evidence that the problem is equivalent to the finite case: Bárány and Steiger construct a probability distribution with $e_{\frac{n-2}{2}}(\mu, n) = \Omega(n \log n)$, based on the original lower bound construction of Straus. The same method can be applied to any recursive lower bound construction of sets $(S_r)_{r=1}^{\infty}$ of sets with many halving edges that is based on local replacements (i.e., if each S_{r+1} is obtained by replacing each point in S_r by a tiny point cloud). In particular, based on Tóth's construction, one can construct a measure μ with $e_{\frac{n-2}{2}}(\mu, n) \geq ne^{\Omega(\sqrt{\log n})}$.

4.3. Points on Convex or Algebraic Curves. For a planar set S of n points in convex position, $e_k(S) = n$ for all $0 \leq k \leq n-2$. Alt, Felsner, Hurtado, and Noy [**18**] extended this and showed that if S is contained in a fixed collection $\mathcal{C}$ of at most B convex curves (bounded or unbounded), for some constant B, then $e_k(S) = O(n)$ for all k, with a constant depending on B. It follows that a linear upper bound also holds true if $\mathcal{C}$ consists of a bounded number of bounded-degree algebraic curves, since each such curve can be cut up into a bounded number (depending on the degree) of convex curves.

It seems reasonable to expect that this can be extended to higher dimensions, e.g., to points lying on a bounded number of bounded-degree algebraic varieties, but to our knowledge, no concrete general results of this kind are known.

4.4. Dense Point Sets. A set of n points in $\mathbf{R}^d$ is called *dense* if the ratio of the largest over the smallest distance between any two points is $O(n^{1/d})$. Edelsbrunner, Valtr, and Welzl [**59**] showed that for dense n-point sets in $\mathbf{R}^2$,

$$e_{1/2} = O(\sqrt{n} e_{1/2}^{(2)}(\sqrt{n})),$$

where the implicit constant in depends on the constant in the definition of density.

In particular, any general bound $e_{1/2}^{(2)}(n) = O(n^{1+c})$ implies a bound of $O(n^{1+c/2})$ for dense point sets. If the number of halving edges were maximized by dense point sets, bootstrapping would lead to $e_{1/2}^{(2)}(n) = O(n \operatorname{polylog} n)$, contradicting Tóth's lower bound.

For dimension $d \geq 3$, Edelsbrunner et al. used the k-set polytope to obtain the improved bound

$$e_{1/2}(S) = O(n^{d-2/d})$$

for dense sets of n points in $\mathbf{R}^d$, $d \geq 3$.The proof proceeds along the following lines:

(1) Assume that S is a dense set of n points in $\mathbf{R}^d$, $n-d$ even, and set $j := (n-d)/2$ and $k := j+1$. Every j-facet σ of S gives rise to a facet $F(\sigma)$ of $P_k(S)$, and since $k - j = 1$, $F(\sigma)$ is just a translated copy of $|\sigma|$. Therefore, the total $(d-1)$-dimensional area of all j-facets is bounded from above by the $(d-1)$-dimensional surface area of $P_k(S)$.

(2) The homothetic copy $\frac{1}{k}P_k(S)$ is contained in the convex hull of S. Therefore, the projection of $\frac{1}{k}P_k(S)$ onto any coordinate hyperplane is contained in the convex hull of the corresponding projection of S, and hence, by density, has $(d-1)$-dimensional area at most $O(n^{\frac{d-1}{d}})$. The total $(d-1)$-dimensional surface area of a convex body is at most two times the sum of

the $(d-1)$-dimensional areas of its projections onto the coordinate hyperplanes. Therefore, the $(d-1)$ dimensional surface area of $P_k(S)$ is "not too large", namely $O(k^{d-1}n^{1-1/d})$. By the first step, the same holds for the total area of all j-facets.

(3) On the other hand, any collection of "many" $(d-1)$-dimensional simplices spanned by points from a dense set necessarily has "large" total $(d-1)$-dimensional area. (The precise statement and the proof of this lemma are somewhat technical, see [**59**] for the details.) Therefore, if there were too many j-facets of S (more than $Cn^{d-2/d}$ for some suitable constant C), then their total area would have to be too large, i.e. would exceed the bound derived in the second step.

4.5. Lattice Points and Corner Cuts. The k-sets of the infinite set $\mathbf{N}_0^d$, which also go under the suggestive name of *corner cuts*, were investigated by Onn and Sturmfels [**112**] in connection with computational commutative algebra. They showed that the set $P_k(\mathbf{N}_0^d) = \operatorname{conv}\{\sum_{x\in X} x : X \subset \mathbf{N}_0^d \text{ and } |X| = k\}$ (which is no longer a bounded polytope) is a convex polyhedron, which they name the *corner cut polyhedron*, and that this polyhedron equals the so-called state polyhedron for a certain kind of ideals in the polynomial ring $K[x_1, \ldots, x_d]$, K any infinite field (namely, vanishing ideals of k generic points in affine d-space over k). As a consequence, the k-sets of $\mathbf{N}_0^d$ are in one-to-one correspondence with the reduced Gröbner bases of such ideals.

Apart from this algebraic connection, corner cuts are a very natural special instance of the k-set problem. Onn and Sturmfels prove an upper bound of $O(k^{2d\frac{d-1}{d+1}})$ for the number $a_k(\mathbf{N}_0^d)$ of corner cuts of cardinality k, in any fixed dimension d. It is not hard to see that all corner cuts of size k are in fact k-sets of the finite set $\{u \in \mathbf{N}_0^d : \prod_i (1+u_i) \leq k\}$ of cardinality $O(k(\log k)^{d-1})$, so one could apply the general k-set bounds, and this would would already lead to an improvement. However, such general methods do not do justice to corner cuts, because of the massive affine dependencies and the density-like volume properties of lattice points. In the plane, Corteel, Rémond, Schaeffer, and Thomas [**52**] showed that $a_k(\mathbf{N}_0^2) = \Theta(k \log k)$. In [**141**], a general upper bound of $a_k(\mathbf{N}_0^d) = O(k^{d-1}(\log k)^{d-1})$ was shown for any fixed dimension d, and a lower bound of $a_k(\mathbf{N}_0^d) = \Omega(k^{d-1} \log k)$ was derived by a simple lifting argument from the planar case. The remaining polylogarithmic gap between upper and lower bounds seems to be due to the simplemindedness of this lifting argument.

An interesting related problem, posed by Onn and Sturmfels [**112**], is whether given $v \in \mathbf{N}_0^d$ and an integer k, we can determine in polynomial time (in k and d) if v is a vertex of the corner cut polyhedron $P_k(\mathbf{N}_0^d)$. The challenge here is that the dimension is considered part of the input (if the dimension is fixed, we can afford to enumerate all the corner cuts).

4.6. Non-General Position. The affine dependencies among the lattice points lend a rather different flavor to corner cuts, compared to the k-set problem for point sets in general position. Kupitz [**91**], Kupitz and Perles (unpublished), and Perles and Pinchasi [**118**] studied a number of questions concerning the separation of point sets in non-general position by spanned hyperplanes.

Let S be a set of n points in $\mathbf{R}^d$ that affinely span $\mathbf{R}^d$. Kupitz showed that for any given k, if n is sufficiently large with respect to d and k, then there is a hyperplane spanned by points in S that contains at least k points of S on either side. In dimension $d = 3$, $n = 4k + 1$ points are sufficient and sometimes necessary; in general dimension, roughly $\lfloor \frac{3}{2}d \rfloor k$ points are sufficient and roughly $2\lceil d/2 \rceil k$ points are sometimes necessary.

A different question is whether there is a k-hyperplane, i.e., a hyperplane spanned by S with exactly k points of S on one side. In dimension $d = 2$, Perles and Pinchasi showed that if $n \geq 2k + 2$, then S has a k-line or a $k + 2$-line.

5. Lower Bounds

The best available lower bound for the number of halving edges is due to Tóth [**137**], who constructed, for every even n, a set of n points in the plane with at least

$$ne^{\Omega(\sqrt{\log n})} \tag{2}$$

many halving edges. Recently, Nivasch [**109**] simplified and streamlined Tóth's construction, which also results also in a better implicit constant in the Ω-notation in the exponent. Specifically, Nivasch works in the dual setting and constructs arrangements of n lines with $\Omega(ne^{\sqrt{\ln 4}\sqrt{\ln n}}/\sqrt{\ln n})$ vertices at the middle level; since his article appears in the present volume, we only briefly outline the idea behind the construction, in the primal setting. We remark that prior to Tóth's work, a lower bound of the form (2) had been obtained by Klawe, Paterson, and Pippenger [**87**] for the middle level in pseudoline arrangements. No better lower bound is known for levels in arrangements of s-intersecting curves, $s > 1$.

The basic reasoning for the lower bound constructions is the following. Consider a set S of n points in the plane. If we "double" each point $p \in S$, i.e., if we replace p by two points p_1 and p_2 sufficiently close to p, then we also double the number of halving lines. Similarly, if we replace each point p by the same number a of collinear points sufficiently close to p then also the number of halving edges in the resulting point set is at least a times the original number, see Figure 17 (if we insist on general position, we can afterwards apply a small perturbation that does not destroy any halving edges). Observe that since the number of halving edges increases by the same factor as the number of points, such a recursive construction yields a linear lower bound.

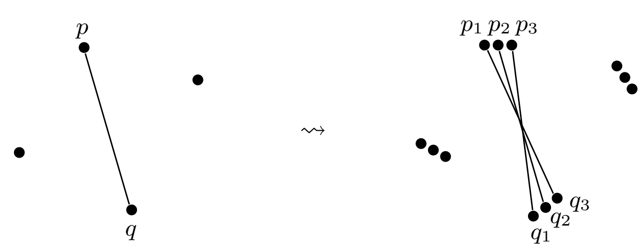


FIGURE 17. Replacing each point by three collinear ones also triples the number of halving edges .

We can improve on this as follows. Select a subset $P \subseteq S$ of $n/2$ points and for each $p \in P$ a halving edge $e(p)$ incident to p such that $e(p) \neq e(p')$ for all $p \neq p'$ in P (this can be done greedily). Now double all the points in S. If $p \in P$ and $e(p) = pq$, we choose the points p_1 and p_2 so that their midpoint is p and the line they span passes through the original point q. For $q \notin P$, we choose the points q_1 and q_2 so that their midpoint is q and that they span a line that does not pass close to any of the original points. Then each original halving edge gets doubled, and additionally, for each $p \in P$, the line through p_1p_2 is also a halving edge of the resulting point set S', see Figure 18. This yields $|S'| = 2n$ and $e_{1/2}(S') \geq 2e_{1/2}(S) + n/2$. This recursion yields an $\Omega(n \log n)$ lower bound.

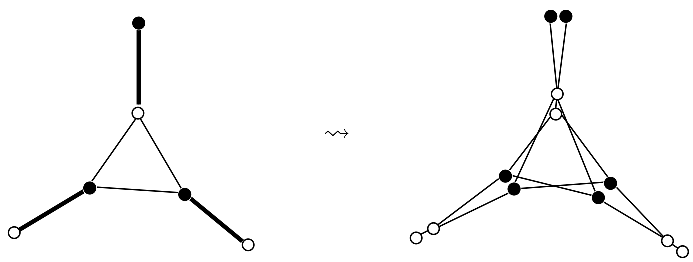


FIGURE 18. A sketch of the replacement scheme for the $\Omega(n \log n)$ lower bound. The points in P are drawn in white, and the selected associated halving edges in bold .

Tóth's improvement is based on the following idea. Suppose that we replace each point in $p \in S$ by a small "arithmetic progression" of length a (equally spaced collinear points $p_1, \ldots, p_a$) and that the directions of all arithmethic progressions are parallel, say horizontal (though not necessarily with the same spacing). Then, as before, each halving edge pq of S is replaced by a halving edges $p_1q_a, p_2q_{a-1}, \ldots, p_aq_1$; moreover, all these halving edges intersect in a single point, and introducing one additional point u slightly to the left to this intersection and one point v to the left of the two lines up_a and uq_a, we can create $2a$ halving edges instead of a, see Figure 19. We would like to do this for all original halving edges pq simultaneously.

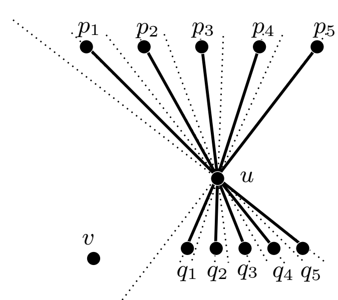


FIGURE 19. The idea for Tóth's construction .

Unfortunately, the points u and v that we introduce for one halving edge pq will interfere with other halving edges, and we have to repair this by introducing further additional points. In order to keep the total number of necessary additional points under control, the main trick is to maintain a *subset* H of halving edges of S (and introduce points u and v as above and the necessary repair points only for these edges). When done carefully, the total number of additional points needed is some constant c times the number $|H|$ of edges. In order to have the number of edges increase by a larger factor than the number of vertices, one can choose $a = 2c|H|/|S|$. We refer to the papers by Tòth and by Nivasch for the details.

5.1. k-Sensitive Lower Bounds. As in the case of upper bounds, lower bounds for the number of halving edges can be turned into lower bounds for k-edges:

PROPOSITION 5.1. *If there are exists a set of $2k+2$ points in the plane with $k \cdot f(k)$ halving edges, then for every $n \geq k$, there exist n-point sets with $e_k \geq \lfloor \frac{n}{2k+2} \rfloor k \cdot f(k) \sim n\, f(k)/2$.*

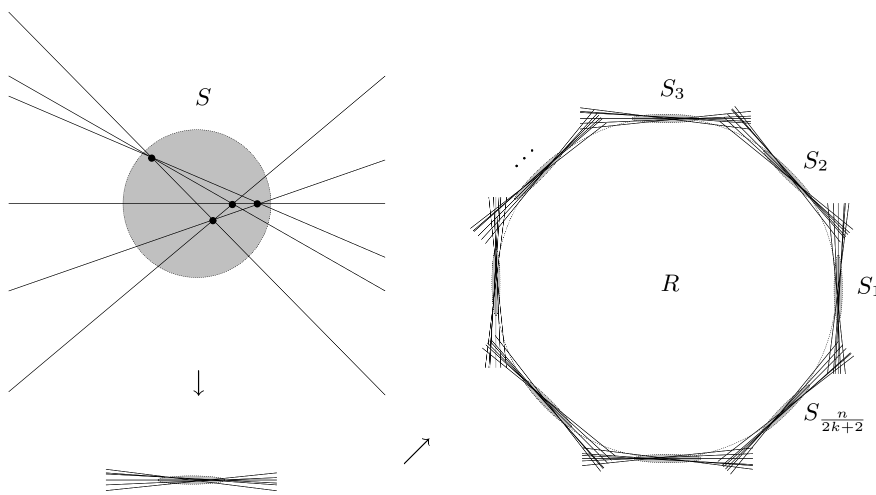


FIGURE 20. Many halving edges yield many k-edges.

PROOF. Let S be the set of $2k+2$ points in the plane. Given any $\varepsilon > 0$, we can apply an affine transformation that does not change the combinatorial structure of S and in particular leaves the number of halving edges invariant, so that the image of S under the transformation is tiny (has diameter at most ε) and flat (all the lines spanned by pairs of points in in the image are ε-close in slope to a given direction). Pick $\lfloor \frac{n}{2k+2} \rfloor$ distinct points on the unit circle (or any other strictly convex curve) and place one tiny flat affine copy S_i of S, $i = 1, \ldots, \lfloor \frac{n}{2k+2} \rfloor$ at each of the points in such a way that the slopes in each copy are ε-close to the tangent to the curve at that point; if n is not divisible by $2k+2$, then a set R of $n-(2k+2)\lfloor \frac{n}{2k+2} \rfloor$ points are placed near the center of the circle. If ε is sufficiently small, then by convexity, the halving edges within each copy S_i are k-edges of the set $S := S_1 \cup \ldots \cup S_{\lfloor \frac{n}{2k+2} \rfloor} \cup R$ (see Figure 20), and the statement follows. □

5.2. From the Plane to Higher Dimensions. Seidel (see Edelsbrunner [**58**]) showed that a lower bound in the plane can be lifted to higher dimensions:

PROPOSITION 5.2. *Suppose that for all sufficiently large n, there are planar n-point sets with at least $n \cdot f(k)$ many $\lceil k/2 \rceil$-edges. Then for any fixed dimension d, there are n-point sets in $\mathbf{R}^d$ with $e_k \geq \Omega(nk^{d-2} f(k))$.*

Proof. Let S' be a set of $n - 2\lfloor k/2 \rfloor - d - 2$ points in $\mathbf{R}^2$ with $\Omega(nf(k))$ many $\lceil k/2 \rceil$-edges. By applying a suitable affine transformation, we may assume that all lines spanned by S' are very close to a fixed line ℓ that passes through the origin o. We embed $\mathbf{R}^2$ into $\mathbf{R}^d$. Furthermore, let $V \subseteq \mathbf{R}^{d-1}$ be a configuration of $2\lfloor k/2 \rfloor + d - 2$ points that are perfectly centered around the origin in the sense of Observation2.6. Thus, every $d-2$-dimensional flat ρ spanned by $d-2$ points of V and the origin o contains exactly $\lfloor k/2 \rfloor$ points of V on either side (within $\mathbf{R}^{d-1}$). We embed $\mathbf{R}^{d-1}$ as the orthogonal complement of ℓ in $\mathbf{R}^d$ and such that no $v \in V$ lies in the copy of the plane $\mathbf{R}^2$. Then for any line ℓ' sufficiently close to ℓ, the span of ℓ' and ρ is a hyperplane in $\mathbf{R}^d$ that still contains exactly $\lfloor k/2 \rfloor$ points of V on either side, see Figure Figure 21. In particular, this holds for any line spanned by S'. Therefore, if pq is a $\lceil k/2 \rceil$-edge of S' and if $\{v_1, \ldots, v_{d-2}\}$ is any $(d-2)$-element subset of V, then then the simplex $pqw_1 \ldots w_{d-2}$ is a k-facet of the set $S := S' \cup V$, since its affine hull partitions the remaining points of V evenly and intersects $\mathbf{R}^2$ precisely in the line through pq. Thus, $e_k(S) \geq e_k(S') \cdot \binom{2\lfloor k/2 \rfloor + d - 2}{d-2} = \Omega(nk^{d-2}f(k))$ (and if desired, a sufficiently small perturbation preserves these k-facets and brings S into general position). ☐

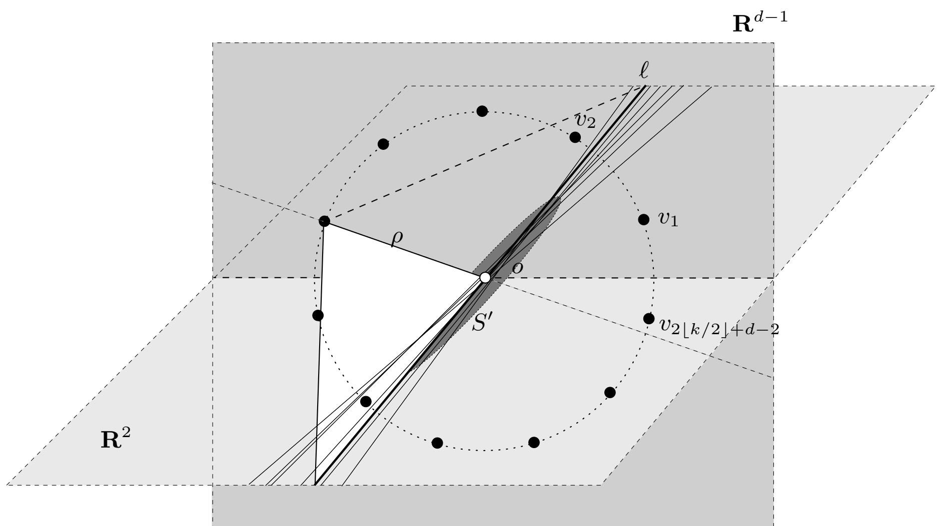


Figure 21. Lifting k-edges to higher dimensions.

We remark that by Proposition 4.1, any neighborly set of n fixed dimension d has $e_k \geq \Omega(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil - 1})$.[11] It would be nice to have a construction that combines the virtues of the previous two, i.e., given planar n-point sets with $e_k \geq nf(k)$, to construct point sets in $\mathbf{R}^d$ with $e_k \geq \Omega(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil - 1} f(k))$.

[11] Alternatively, one can start with a neighborly d-polytope P on $\lfloor n/(k+1) \rfloor$ vertices and construct S by replacing each vertex v of P by $k+1$ points equally spaced along a unit segment normal to P at v. The number of facets of P (or 0-facets of the set V of vertices) is $\Omega(|V|^{\lfloor d/2 \rfloor}$. Each facet of P gives rise to $\binom{k+d-1}{d-1}$ many k-facets of S, so $e_k(S) \geq \Omega((n/k)^{\lfloor d/2 \rfloor} k^{d-1})$.

6. Upper Bounds for Halving Facets in All Dimensions

Almost all known proofs of upper bounds exploit only one simple local property of the hypergraph of halving facets of a finite point set S, usually referred to as the "antipodality" or "interleaving" property: If ρ is a $(d-2)$-dimensional simplex spanned by points of S, and if h is a hyperplane that passes through ρ, then the affine hull of ρ divides h into two "half-hyperplanes". Roughly speaking, the interleaving property means that as we rotate h about ρ, we encounter the halving facets incident to ρ alternatingly in in one half-hyperplane and in the other. Below we give a more formal definition.

6.1. The Interleaving Property.

Definition 6.1. A *geometric hypergraph* in $\mathbf{R}^d$ is a pair (S, T), where S is a finite set of points in general position, and T is a collection of simplices spanned by points from S. The elements of T are also called hyperedges. A geometric hypergraph is called k-uniform if all hyperedges have k vertices, i.e., if all hyperedges are simplices of dimension $k-1$. For a 2-uniform geometric hypergraph in $\mathbf{R}^2$ we drop the prefix "hyper" and just speak of a geometric graph and its edges.

We will often denote an (unoriented, uncooriented) simplex by an unordered list of its vertices. Thus, $p_1 \cdots p_k = \operatorname{conv}\{p_1, \ldots, p_k\}$, with the understanding that the points are affinely independent.

Definition 6.2 (Interleaving Property). A d-uniform geometric hypergraph (S, T) in dimension $\mathbf{R}^d$ is called *interleaving* if the following holds for any $d-1$ points $p_1, \ldots, p_{d-1} \in S$: Whenever $a, b \in S$ are two distinct points such that both $ap_1 \cdots p_{d-1}$ and $bp_1 \cdots p_{d-1}$ are simplices in T, then there is a third point $c \in S$ such that $cp_1 \cdots p_{d-1} \in T$ and such that the triangle abc intersects the affine hull of $p_1 \cdots p_{d-1}$ (see Figure 22).

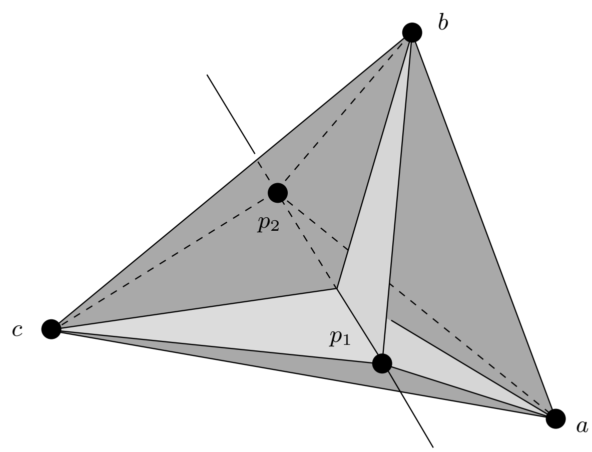


Figure 22. The interleaving property in three dimensions.

Lemma 6.3. *Let S be a finite set of n points in general position in $\mathbf{R}^d$, $n-d$ even, and let T be the set of halving facets of S. Then (S, T) is an interleaving d-uniform geometric hypergraph.*

Proof. Fix a $(d-2)$-dimensional simplex $\rho = p_1 \dots p_{d-1}$ spanned by points in S and suppose that $a\rho$ and $b\rho$ are two distinct halving facets incident to ρ. Consider a hyperplane h rotating about ρ from a to b. Initially, while h passes through a, both open halfspaces determined by h contain exactly $k := \frac{n-d}{2}$ points from S. Immediately after we start rotating, one of the open halfspaces also contains a and therefore $k+1$ points. On the other hand, when we reach b, this halfspace will again contain k points. Moreover, the number of points in the halfspace only changes when h passes over some point of S, and then it changes by ± 1, by general position. Therefore, at some moment during the rotation, h must pass over a point c such that the count changes from $k+1$ to k. That point is of the desired kind. $\square$

Remark 6.4. The hypergraph of halving facets actually satsifies the following, stronger property: Let $\rho = p_1 \dots p_{d-1}$ be a $(d-2)$-dimensional simplex spanned by S, and let h be a hyperplane passing through ρ but not containing any other point of S. Since the number of remaining points is is odd, one of the two open halfspaces contains more of them than the other. Consider the halving facets incident to ρ. Suppose there are m of them whose remaining vertex lies in the smaller halfspace. Then there are exactly $m+1$ for which it lies in the larger halfspace, see Figure 23. This can be be proved by the same continuous rotation argument as before. Moreover, one can show that this stronger interleaving property characterizes the hypergraph of halving facets.

As a consequence, any $(d-2)$-dimensional simplex $\rho = p_1 \dots p_{d-1}$ spanned by S is incident to an odd number of halving facets, in particular to at least one. Since each halving facet is incident to exactly d simplices ρ of dimension $d-2$, it follows that $e_{1/2}(S) \geq \binom{n}{d-1}/d = \Omega(n^{d-1})$.

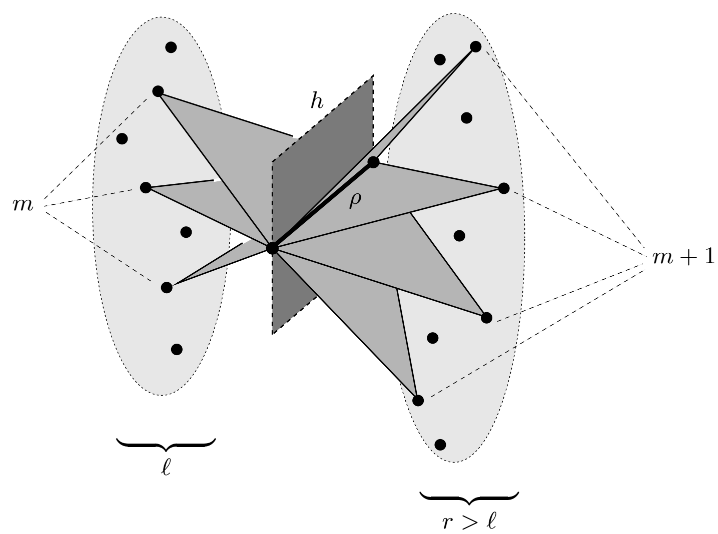


Figure 23. A stronger interleaving property.

6.1.1. *The General Strategy.* Very roughly speaking, most existing proofs for upper bounds for the number of simplices in an interleaving geometric hypergraph (S, T) proceed along the following lines (or can, with hindsight, be interpreted in this way).

0. Find a simple kind of "conflict" or "crossing" determined by a bounded number of points from the ground set S. Examples of crossings that have been successfully considered include
   (a) pairs (ℓ, σ) where ℓ is a line, $\sigma \in T$, and ℓ intersects the relative interior of σ in a single point (there are several variants as to how the line may be determined by points from the ground set, see below);
   (b) pairs (σ, τ) of *crossing* simplices in T, i.e., σ and τ are vertex-disjoint and their relative interiors intersect;
   (c) "pinched crossings", i.e., pairs (σ, τ) of simplices in T that share a single vertex and whose relative interiors have a $(d-2)$-dimensional intersection.

(1) Show that if $t = |T|$ is "large" with respect to $n = |S|$ (usually the assumption is $t \geq \Omega(n^\gamma)$ for some suitable $\gamma \geq d-1$), then there are "many" crossings of the given kind (where "many" usually means at least $\Omega(t^a/n^b)$ for suitable constants a, b). Usually, it is enough to show that there is at least one crossing and then to amplify the bound using the Abstract Crossing Lemma 3.2.

(2) Show that for an *interleaving* hypergraph, there cannot be "too many" crossings, after all (where "not too many" usually means at most $O(n^c)$ for some suitable c). A typical example for such a second step is Lovász' Lemma.

Combining the lower bound from Step 1 and the upper bound from Step 2 and solving for t, one obtains $t \leq O(\max\{n^\gamma, n^{\frac{b+c}{a}}\})$, so in order to successfully implement this strategy, both γ and $\frac{b+c}{a}$ should be smaller than d, or indeed smaller than the currently best bound one wishes to improve upon. In order to give some substance to this vague outline, it is best to consider some concrete instances, which we do in the following subsections.

We remark that in most implementations of this strategy, the lower bound in Step 1 is proved for general d-regular geometric hypergraphs, and the interleaving property is only exploited in Step 2. For an exception to this rule, see Lemma 8.1 below.

6.2. Lovász' Lemma. Consider a $(d-1)$-dimensional simplex σ and a line ℓ in $\mathbf{R}^d$. We say that ℓ *crosses* σ if ℓ intersects the relative interior of σ in a single point (this is the generic way such a simplex and a line intersect.) The following lemma takes care of Step 2:

LEMMA 6.5 (Lovász' Lemma). *Let (S, T) be an interleaving d-regular geometric hypergraph on $|S| = n$ points. Then any line ℓ crosses at most $\frac{1}{2}\binom{n}{d-1}$ simplices in T.*

We remark that for the special case of halving facets, there is an exact version of Lovász' Lemma, which we will discuss in Section 10 and which turns out to be equivalent to the Upper Bound Theorem for convex polytopes. For the purposes of upper bounds on the order of magnitude, however, the simpler version above is all we need.

PROOF OF LOVÁSZ' LEMMA. Consider a line ℓ. If it crosses a $(d-1)$-simplex τ, then this is still the case after a small perturbation of ℓ. Therefore, we may assume that ℓ does not contain any points of S and that there is a 2-dimensional plane π

that contains ℓ and that does not intersect any $(d-3)$-dimensional simplex spanned by points in S (pick any 2-plane containing ℓ, and if it is not in general position, slightly perturb π and ℓ with it). Suppose we translate ℓ in parallel within the plane π. If we move sufficiently far away from the original position, the resulting line will no longer intersect the convex hull of S and therefore not cross any simplices at all. Moreover, as we continuously translate ℓ towards that final position, the number of $(d-1)$-dimensional simplices crossed by ℓ changes only when we pass through a $(d-2)$-dimensional simplex, and by the interleaving property, it changes by ± 1 at such a moment. Moreover, there are only $\binom{n}{d-1}$ such $(d-2)$-dimensional simplices alltogether. There are two possible directions in which we can translate ℓ within π. By choosing the direction in which we pass through fewer $(d-2)$-simplices, the bound follows. $\square$

6.3. The Second Selection Lemma. Having Lovász' Lemma 6.5 at our disposal, the question is how to find a line that crosses "many" simplices from a given geometric hypergraph (S, T). Here is the approach developped by Bárány, Füredi, and Lovász [**28**]. (Our presentation is inspired by the exposition in Matoušek's textbooks [**99, 100**].) First project the points from S orthogonally onto a generic hyperplane, which we identify with $\mathbf{R}^{d-1}$. Thus, we obtain a set $\overline{S}$ of n points in general position in $\mathbf{R}^{d-1}$, and the simplices from T project to a family $\mathcal{T}$ of full-dimensional simplices on $\overline{S}$. If we can find a point o in $\mathbf{R}^{d-1}$ that is contained in a certain number m of simplices from $\overline{T}$, then the inverse image of o under the projection is a line that intersects m simplices from T. Thus, applying the following theorem in dimension $d-1$ gives us a line as desired. For convenience, we state the theorem without the shift in dimension:

Theorem 6.6 (Second Point Selection Lemma). *Let S be a set of n points in general position in $\mathbf{R}^d$, and let $\mathcal{T}$ be a family of full-dimensional (i.e., d-dimensional) simplices spanned by S. If we write $|\mathcal{T}| = \alpha\binom{n}{d+1}$, with $\alpha \in (0,1]$, then there exists a point in $\mathbf{R}^d$ that is contained in at least*

$$c\alpha^{s_d}\binom{n}{d+1}$$

simplices of $\mathcal{T}$, where $c = c_d > 0$ and $s_d > 0$ are constants that depend only on the dimension.

In dimension $d = 1$, it is not hard to see that $s_1 = 2$. In dimension 2, the best upper bound on the exponent is $s_2 \lesssim 3$. More precisely, if $\mathcal{T}$ is a set of $\alpha\binom{n}{3}$ triangles on n points in general position in the plane, then there exists a point common to at least $\Omega((\alpha^3/\log^5(n))\binom{n}{3})$ (see Aronov, Chazelle, Edelsbrunner, Guibas, Sharir, and Wenger [**22**]). The current proof of the general case yields a huge exponent $s_d = (4d+1)^{d+1}$, and the best that could possibly be proved by this method is $(d+1)^{d+1}$. A construction due to Eppstein [**63**] shows that $s_2 \geq 2$. Beyond that, the knowledge of lower bounds for the exponent seems very scarce.

For the special instance $\alpha = 1$, which is sometimes referred to as the "First Selection Lemma", various significantly simpler proofs are known. In Section 10, we will present one that is based on k-facets. The strength of the Second Selection Lemma is that the parameter α is not constant; for the bound for the number of halving simplices, we use $\alpha = n^{-1/s_d}$. Together with Lovász' Lemma, this implies:

COROLLARY 6.7. *If* (S,T) *is an interleaving geometric hyperpgraph in* $\mathbf{R}^d$*, then the number of hyperedges satisfies*

$$|T| \leq O\big(n^{d-1/s_{d-1}}\big),$$

with the exponent $s_{d-1} >$ *from the Second Selection Lemma in dimension* $d-1$.

The structure of the proof of the Point Selection Lemma is as follows. Suppose that $\mathcal{T}$, as an abstract hypergraph, contains a copy of the complete $(d+1)$-uniform $(d+1)$-partite hypergraph $K^{(d+1)}(t,\dots,t)$ with t vertices per class, where $t = t(d+1,d)$ is the number in the Colorful Tverberg Theorem 2.4 for $r = d+1$; thus, the current proofs give $t = 4d+1$, and $t = d+1$ is the best we can ever hope for. The theorem tells us that there are some $d+1$ vertex-disjoint simplices $\Delta_1,\dots,\Delta_{d+1}$ contained in that copy and hence in $\mathcal{T}$ whose common intersection is nonempty. Let us call such a $(d+1)$-tuple an *intersecting tuple*.

The existence of such a copy of $K^{(d+1)}(t,\dots,t)$, in fact of many such copies, is guaranteed by the following theorem. Note that one cannot guarantee the existence of a copy of the complete $(d+1)$-uniform hypergraph $K^{(d+1)}(t)$ based on the order of magnitude of $|\mathcal{T}|$ alone (for instance, a complete multipartite hypergraph does not contain a complete one). For this reason, one cannot apply Tverberg's original Theorem and a colorful version had to be invented.

THEOREM 6.8 (Erdős-Simonovits Theorem [**67**]). *Let* d,t *be positive integers, and let* $\mathcal{T}$ *be a* $(d+1)$*-uniform hypergraph on* n *points. Suppose that* $\mathcal{T}$ *has at least* $\alpha\binom{n}{d+1}$ *hyperedges, where* $\alpha \geq Cn^{-1/t^d}$ *for some universal constant* C*. Then* $\mathcal{T}$ *contains at least*

$$\Omega\big(\alpha^{t^{d+1}} n^{(d+1)t}\big)$$

copies of $K^{(d+1)}(t,\dots,t)$*, where the implicit constant depends only on* d *and* t.

(Again, it would be enough to guarantee a single copy, and one could amplify the result by random sampling, but it seems not more difficult to exhibit many copies than a single one.)

For every copy of $K^{(d+1)}(t,\dots,t)$, we get an intersecting $(d+1)$-tuple of simplices in $\mathcal{T}$. Moreover, every such $(d+1)$-tuple of simplices arises from at most $O(n^{(d+1)t-(d+1)^2})$ different copies of $K^{(d+1)}(t,\dots,t)$ (this is the number of ways to complete the $(d+1)^2$ vertices of the simplices to a $t(d+1)$-element subset of S). Thus, we get at least $\alpha^{t^{d+1}} n^{(d+1)^2}$ different intersecting $(d+1)$-tuples of simplices in $\mathcal{T}$ such that each $(d+1)$-tuple of simplices have a common point of intersection.

If every $(d+1)$-tuple of simplices in $\mathcal{T}$ were intersecting, then we could conclude from Helly's Theorem that the intersection of all simplices in $\mathcal{T}$ is nonempty. Instead, we are only guaranteed that some fraction of $\varepsilon = \alpha^{t^{d+1}}$ of the $(d+1)$-tuples are intersecting. A fractional version of Helly's Theorem, due to Katchalski and Liu [**84**], implies that in this case, there is some $\delta > 0$ such that some subset of $\varepsilon|\mathcal{T}|$ simplices from $\mathcal{T}$ have a nonempty intersection. Namely, we can take $\delta = \alpha^{s_d}/(d+1)$, where $s_d := t^{d+1} - 1 = (4d+1)^{d+1} - 1$. This completes our sketch of the proof of the Second Selection Lemma.

THEOREM 6.9 (Fractional Helly Theorem). *For every dimension* d *and every parameter* $1 \geq \varepsilon > 0$*, there exists* $\delta = \delta(d,\varepsilon) > 0$ *such that the following holds: Let* $\mathcal{F}$ *be a family of* N *convex sets in* $\mathbf{R}^d$*. If at least* $\varepsilon\binom{N}{d+1}$ $(d+1)$*-tuples* $(F_1,\dots,F_{d+1})$

of members of $\mathcal{F}$ have a common intersection, then there is a point that lies in δN sets of $\mathcal{F}$. In fact, we can take $\delta = \varepsilon/(d+1)$.[12]

7. Crossings in Dimension 2.

Prior to Dey's [**53**] work, all proofs of upper bounds in the plane explicitly or implicitly relied on Lovász' Lemma (or a dual version for line arrangements). Dey's improvement is based on replacing "conflicts" of the type "a line crossing a halving edge" by crossing pairs of halving edges. The Crossing Lemma provides the lower bound in Step 1 of the general strategy above. To complete the proof, one needs an upper bound for the number of such crossings. More generally, the following holds:

THEOREM 7.1. *If (S,E) is an interleaving geometric graph on n vertices, then the number of crossings is at most $(n/2)^2$.*

Together with the Crossing Number Theorem, this immediately implies:

COROLLARY 7.2. *If (S,E) is an interleaving geometric graph on $|S| = n$ vertices, then the number of edges satisfies $O(n^{4/3})$.*

7.1. Convex Chains.

We present a simplified version of Dey's proof, based on lectures by Sharir. The number of crossings in an interleaving geometric graph can be analyzed using a partition of the edge set into so-called *convex chains*. This partition depends on the choice of an (x,y)-coordinate system for the plane such that no two points in S have the same x-coordinate and no two edges in E have the same slope. The union of the edges in each part will be an x-monotone convex polygonal curve (i.e., the graph of a piecewise linear partial function defined on some interval). This method was introduced (in the dual setting of line arrangements) by Aronov, Agarwal, Chan, and Sharir [**2**].

Let (S,E) be an interleaving geometric graph on $|S| = n$ points. Since the goal is to prove upper bounds in terms of n, we may assume that every point of S is incident to at least one edge from E. Then it follows from the interleaving property that in fact, every point has odd degree. (In particular, since the sum of the degrees is twice the number of edges, it follows that n must be even.)

Let p,q, and r be three points in S such that $x_p < x_q < x_r$ in the chosen coordinate system and such that both pq and qr are edges in E. The we call pq a *left neighbor* of qr and conversely, qr a *right neighbor* of pq. Moreover, a (left or right) neighbor e' of an edge e is called an *upper* or *lower* neighbor of e, respectively, depending on whether e' lies in the upper (w.r.t. to the y-direction) or lower halfplane determined by (the line through) e.

If $pq \in E$ has at least one right upper neighbor edge, then we pick the one closest in slope to pq and declare it the *convex successor* of pq; otherwise, the convex successor is undefined. Similarly, the *convex predecessor* of pq is the upper left neighbor closest in slope, if it exists. A *convex chain* is an inclusion-maximal sequence $e_1, \ldots, e_m$ of edges in E such that e_i is a convex successor of e_{i-1}, $1 \leq i \leq m$, see Figure 24. Concave predecessors, successors, and chains are defined analogously in terms of lower left and right neighbors.

As an immediate consequence of the interleaving property, we have:

[12] The best possible $\delta = 1 - (1-\varepsilon)^{1/(d+1)}$ was proved by Kalai [**81**].

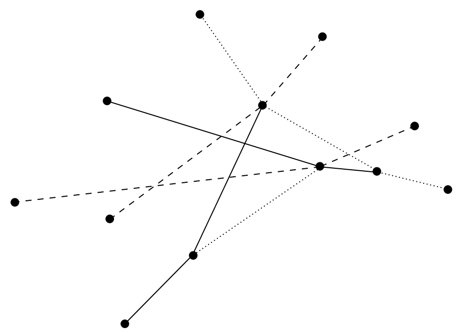


FIGURE 24. An interleaving graph and the partition of its edges into six convex chains (two drawn solid, two dashed, and two dotted).

LEMMA 7.3. *If qr is the convex [concave] successor of pq, then pq is the convex [concave] predecessor of qr, and vice versa. In other words, two distinct edges cannot have the same convex [convcave] successor or predecessor.*

It follows that the convex [concave] chains indeed form a partition of E (if two convex [concave] chains overlap in one edge, they must be the same).

Every convex [concave] chain has two endpoints, a left one and a right one. Conversely, every point $p \in S$ is the endpoint of exactly one convex [concave] chain, namely the chain corresponding to the edge whose slope is maximal in absolute value among all the edges incident to p. Therefore, there are exactly $n/2$ convex [concave] chains.[13]

Given a crossing between two edges e and e' in E, we can interpret it as a crossing between the convex chain through e and the concave chain through e', or vice versa. Since a convex chain and a concave chain can intersect at most twice, Theorem 7.2 follows.

REMARKS 7.4.

(1) In the dual setting, Dey's result bounds the number of vertices at the middle level in an arrangement of lines in the plane. Tamaki and Tokuyama [**135**] generalized Dey's bound to arrangements of pseudolines, and a simpler proof for this was given by Sharir and Smorodinsky [**124**].

(2) Theorem 7.2 is tight for interleaving geometric graphs [**128**]; the construction is based on a construction of Erdős of a collection of n points and n lines in the plane with $\Omega(n^{4/3})$ incidences between the points and the lines. Thus, in order to prove better upper bounds for the number of halving edges, one has to find and exploit further properties (unless, of course, there really are point sets with $\Omega(n^{4/3})$ halving edges).

7.2. A Refined Analysis for k-Edges in the Plane. Aronov, Adrzejak, Har-Peled, Seidel, and Welzl [**19**] refined the analysis of the number of crossings for k-edges and obtained the following results for any set S of n points in general position in the plane.

[13] For the graph of halving edges of a set n points, it follows from the stronger interleaving property mentioned in Remark 6.4, that each convex chain starts at one of the $n/2$ leftmost vertices and ends at one of the $n/2$ rightmost ones.

Theorem 7.5 (Halving-Edge Crossing Identity). *Assume that n is even. Let $\deg_{1/2}(p)$ denote the number of halving edges incident to a point p (which is always odd), and let $X_{1/2}$ be the number of (proper) crossings of halving edges. Then*

$$X_{1/2} + \sum_{p \in S} \binom{(\deg_{\frac{1}{2}}(p)+1)/2}{2} = \binom{n/2}{2}$$

The following corollary was independently obtained by Pach and Solymosi [**114**]:

Corollary 7.6. *The number of halving edges is minimized (namely, equal to $n/2$) iff the halving edges pairwise cross each other.*

Theorem 7.7 (k-Edge Crossing Identity).
For $0 \leq k < (n-2)/2$, let $\overrightarrow{\deg}_k(p)$ denote the out-degree *of p in the directed graph of k-edges, i.e., the number of k-edges emanating from p, and let X_k denote the number of crossings between k-edges. Then*

$$X_k + \sum_{p \in S} \binom{\overrightarrow{\deg}_k(p)}{2} = e_{<k}.$$

Theorem 7.8 (j-k-Edge Crossing Identity). *For $0 \leq j < k < (n-2)/2$, let $X_{j,k}$ denote the number of crossings between j-edges and k-edges. Then*

$$X_{j,k} + \sum_{p \in S} \overrightarrow{\deg}_j(p)\Big(\overrightarrow{\deg}_k(p) - 1\Big) = 2e_{<j}.$$

As a corollary, one obtains the following bound for sums $\sum_{k \in K} e_k$, which is better than summing up the individual bounds. We will see an application of such sums in Section 9.

Corollary 7.9. *For any set of n points in general postion in the plane and any set $K \subseteq [0, \ldots, \lfloor \frac{n-3}{2} \rfloor]$ of indices,*

$$\sum_{k \in K} e_k(S) = O(n \sqrt[3]{|K| \textstyle\sum_{k \in K} k}).$$

Proof of the corollary. Let E be the set of all k-edges with $k \in K$. Using the bound $e_{<k} \leq nk$, we see that the total number of crossings in the geometric graph (S, E) is

$$\sum_{k \in K} X_k + \sum_{\substack{j,k \in K \\ j<k}} X_{j,k} \leq n \sum_{k \in K} k + n \sum_{j \in K} \sum_{\substack{k \in K \\ k > j}} j \leq n|K| \sum_{k \in K} k.$$

Applying the Crossing Lemma yields the desired bound. $\square$

7.2.1. *Continuous motion.* Aronov, Andrzejak, Har-Peled, Seidel, and Welzl proved their results using continuous motion arguments. The idea is that the above identities are easy to check if all the points are in convex position. Moreover, any set of n points can be brought into convex position by a continuous motion of the points. (A labelled set of n points in the plane corresponds to a point in $\mathbf{R}^{2n}$, and a continuous motion to a continuous path in $\mathbf{R}^{2n}$.) Such a continuous motion can be chosen so that there are only finitely many discrete times at which a single triple of points is collinear. (A collinearity between a given triple corresponds to a hypersurface in $\mathbf{R}^{2n}$, given by a polynomial equation—the vanishing of a determinant—, and a path can be chosen so as to avoid any singularities of or intersections between such hypersurfaces, and such that each intersection with a hypersurface is transverse.)

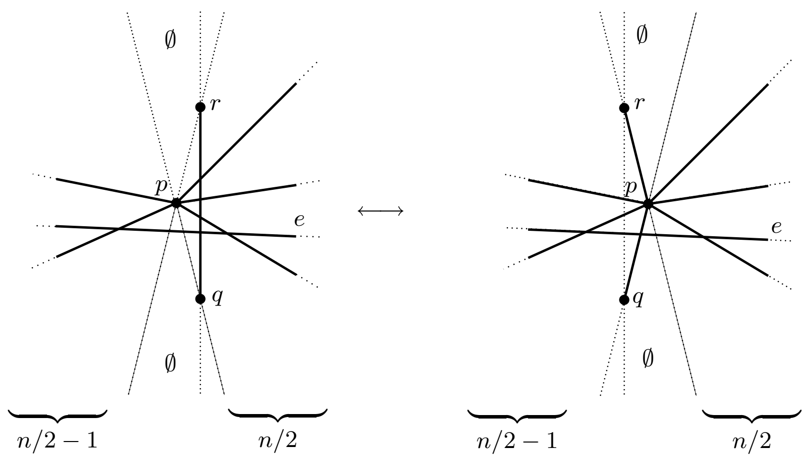


FIGURE 25. The mutations for halving edges during a continuous motion.

In between such collinearity changes, the combinatorial type of the point set and in particular all quantities involved in the above identities remain unchanged. Moreover, observing what happens in an ε-interval around a collinearity, one sees that in the case of halving edges, for example, the only mutations that have any effect on the quantities involved are those for which at the moment of collinearity of the triple, there are $n/2-1$ points on either side of the line through the three points, see Figure 25.

In such a mutation, as we pass from the picture on the left to the picture on the right, the degree of the central point p increases by 2t. At the same time, we lose the crossing between the edge qr and the halving edges emanating from p to the right. Crossings between qr and any other edge e are "distributed" to the new edges pq and pr. All other degrees and crossings remain unaffected. By the strong interleaving property (Remark 6.4), the number of halving edges emanating from p to the right, into the larger halfspace, equals $(\deg_{\frac{1}{2}}(p)+1)/2$. Thus, the number of crossings that we lose is exactly balanced by the increase $\binom{(\deg_{\frac{1}{2}}(p)+2+1)/2}{2} = \binom{(\deg_{\frac{1}{2}}(p)+1)/2}{2}$.

7.2.2. *k-Edge Curves.* The identities for k-edges can also be proved by continuous motion. Alternatively, they can be proved by using a bijection between crossings and bitangents for locally convex plane curves. First, one defines a decomposition of the set of k-edges similar to the convex chain decomposition for halving edges, but independent of the coordinate system (only depending on the orientation): For a directed k-edge $\overrightarrow{pq}$, define its successor by rotating the line through p and q counterclockwise about the head q until we reach the next k-edge. The difference to the convex chain decomposition used above is that we do not stop when we reach the vertical direction. Again, for every successor, there is a unique predecessor, and we get a partition of the set of k-edges into several locally convex polygonal curves, see Figure 26. The crucial property of the decomposition is the following: While rotating a directed line ℓ from a k-edge to its successor, there are always at most k points to the right of ℓ.

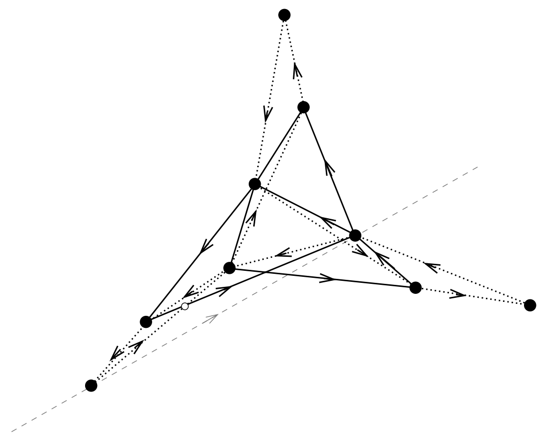


FIGURE 26. The 3-edge curve Γ_3 of this set of nine points has two components (one solid, one dotted). One of self-intersections is marked by a white circle, and the corresponding bitangent is drawn as a dashed grey line (and corresponds to a 2-edge).

The resulting multicomponent curve Γ_k has self-intersections either at crossing between edges, or at original points. Moreover, by the aforementioned property of the decomposition, any two branches of the curve passing through a point intersect tranversely (there are no self-tangencies). If there are r branches of the curve passing through p, i.e., if $\overrightarrow{\deg}_k(p) = r$, then we interpret this as $\binom{r}{2}$ self-intersections. Thus, we see that the left-hand-side of the Crossing Identity for k-edges counts precisely the number of self-intersections of Γ_k.

For every traverse intersection between two branches of of a locally convex plane curve, there is a unique bitangent to the two braches. In the case of the k-edge curve, this tangent is spanned by two original points, by general position, and has strictly less than k points to its right, and hence corresponds to a j-edge with $j < k$.

Conversely, given such a bitangent between two branches of a locally convex curve, we can reconstruct a unique crossing for it, unless the two branches "bend away" from each other before they cross, see Figure 27. In the case of the k-edge curve Γ_k, $k < (n-2)/2$, this is impossible, because otherwise there would be two open halfspaces with parallel bounding lines, each of them tangent to one of the branches, such that the union of the halfspaces covers the plane yet each of them contains at most k points, a contradiction. Thus, the $(< k)$-edges are in one-to-one correspondence with the self-intersections of Γ_k.

REMARK 7.10. In higher dimensions, one can define a notion of "convex neighbor" of a k-facet $\sigma = p_1 \dots p_d$ with respect to a ridge $\rho = p_1 \dots p_{d-1}$ (thus, a k-facet has d neighbors). This yields a decomposition of the set of k-facets of a finite point set into locally convex closed "hypersurfaces". However, these hypersurfaces are no longer (the images of) topological manifolds (in the sense that a closed plane curve is the image of a circle), because they can be "pinched" at vertices. Instead, each component of the k-facet hypersurface is an immersed "pseudomanifold" (these are

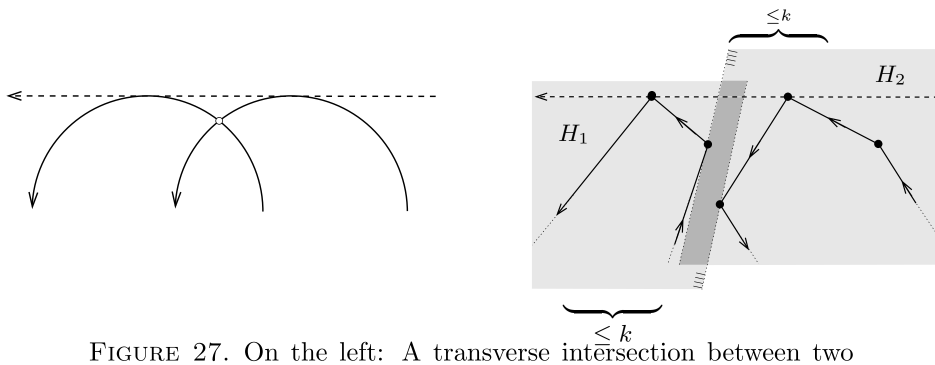


FIGURE 27. On the left: A transverse intersection between two branches of a locally convex curve corresponds to a bitangent. On the right: Conversely, in the case of the k-edge curve Γ_k, $k < n/2$, every bitangent must correspond to a crossing.

defined by the property that each ridge is incident to two facets). It would be interesting to prove higher-dimensional analogues of the crossing identities.

8. Improvements in Three And Four Dimensions

As remarked above, while the combination of Lovász' Lemma and the Second Selection Lemma does prove nontrivial upper bounds of the form $O(n^{d-\varepsilon_d})$ for the number of halving facets in any fixed dimension, the method based on the Colorful Tverberg Theorem cannot deliver a constant ε_d larger than d^{-d}.

8.1. The Current Record in $\mathbf{R}^3$. Aronov, Chazelle, Edelsbrunner, Guibas, Sharir, and Wenger [**22**] proved a planar version of the Second Selection Lemma using more elementary methods. By applying Lovász' Lemma as before, this results in a bound of $e^3_{1/2}(n) = O(n^{8/3} \log^{5/3}(n))$ for the number of halving triangles of n points in $\mathbf{R}^3$.

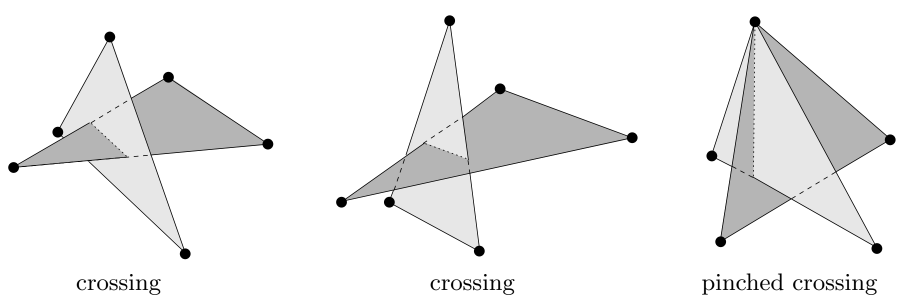


FIGURE 28. The two possible ways in which vertex-disjoint triangles can cross, and a pinched crossing.

Dey and Edelsbrunner removed the polylogarithmic factor, by abandoning the projection step and directly working with crossing pairs of triangles, where two triangles τ_1 and τ_2 are said to cross if they are vertex disjoint but their relative interiors intersect, see Figure 28. They show that if (S,T) is a geometric hypergraph in $\mathbf{R}^3$ with $|S| = n$ points and $t := |T| > \frac{3}{2}n^2$, then there are two triangles τ_1 and τ_2 of T that cross. By the Abstract Crossing Lemma, it follows that for $t > 2n^2$, there are at least $\Omega(t^4/n^6)$ pairs of crossing triangles in T.

On the other hand, if τ_1 and τ_2 cross, then some edge $e = pq$ of τ_1 intersects the relative interior of τ_2 in a single point (or vice versa). Moreover, any segment pq is an edge of at most t triangles of T. Therefore, if there are x pairs of crossing triangles, then there is a segment that crosses at least x/t triangles in T. So far, the argument holds for any collection of triangles. If (S, T) is interleaving, then by Lovász' Lemma no line (and therefore, no segment) crosses more than $O(n^2)$ triangles of T. It follows that $\Omega(|T|^3/n^6) \le x/t \le O(n^2)$ i.e., $|T| = O(n^{8/3})$.

The currently best upper bound for the number of halving triangles in three dimensions, due to Sharir, Smorodinski, and Tardos [**125**] makes more extensive use of the interleaving property. The first difference is to consider "pinched crossings", i.e., pairs (τ_1, τ_2) of triangles whose relative interiors intersect and which share exactly one vertex, see Figure 28. Note that if τ_1 and τ_2 are pinched, then there is an edge $e = pq$ of one of them, say of τ_1, that intersects the relative interior of the other triangle, τ_2. Moreover, a given pair (pq, abc) such that the segment pq intersects the interior of the triangle abc can be completed in at most three ways to a pinched pair of triangles: The other triangle must be one of apq, bpq, cpq. Thus, if a geometric hyperpgraph (S, T) on $|S| = n$ points in $\mathbf{R}^3$ contains x pinched crossings, then some segment pq intersects at least $\Omega(x/n^2)$ triangles of T. Thus, by Lovász' Lemma, if (S, T) is interleaving then $x \le O(n^4)$.

Furthermore, Sharir et al. also make use of the interleaving property for the second step of exhibiting many pinched crossings.

Lemma 8.1. *If (S, T) is an interleaving geometric hypergraph in $\mathbf{R}^3$ with $|S| = n$ points and $t := |T|$ triangles, then the number x of pinched crossings of triangles in T satisfies*

$$x \ge \Omega(t^2/n) - O(tn).$$

Consequently, either $t = O(n^2)$, or $x \ge \Omega(t^2/n)$. Balancing this with the upper bound for x yields:

Theorem 8.2. *If (S, T) is an interleaving geometric graph in $\mathbf{R}^3$ on $|S| = n$ points, then the number of triangles satisfies*

$$|T| = O(n^{5/2}).$$

In particular, this holds for the family of halving triangles of S.

The idea for the proof of Lemma 8.1 is to consider the radial projection of the triangles incident to a each point p onto a 2-dimensional plane. This projection will almost be a geometric graph in the plane (with the exception that the edges can also be semiinfinite rays), and a crossing in this graph will correspond to a pinched crossing of triangles with apex p.

More precisely, we choose a generic (x, y, z)-coordinate system and a horizontal plane π (one that is parallel to the xy-plane) that passes below all points of S. For each triangle $\tau \in T$, we classify its vertices as "lower", "middle", and "upper", respectively, according to their z-coordinates. For each point $a \in S$, let U_a be the set of triangles from T that have a as their upper vertex, and let M_a be the set of triangles that have a as their middle vertex. We project the triangles in $U_a \cup M_a$

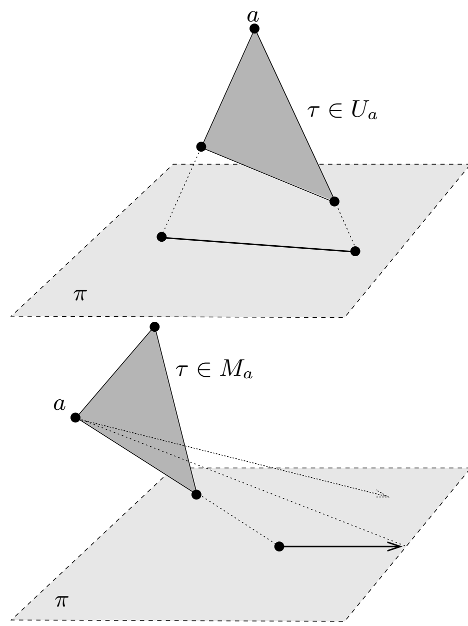


FIGURE 29. The radial projections of triangles in U_a and in M_a, respectively, onto π.

radially from a onto the plane π, see Figure 29. Triangles in U_a project to segments, and triangles in M_a project onto semiinfinite rays in π. An endpoint of such a ray or segment corresponds to the intersections of π with the line through a and some other point $b \in S$ that lies below a.

We regard the collection of rays and segments obtained for one particular a as a generalized geometric graph G_a. Let $r_a = |M_a|$ denote the number of rays in G_a, let $t_a = |U_a| + |M_a|$ the number of edges, i.e., of rays and segments, in G_a, and let n_a be the number of vertices of G_a, which equals the number of points from S that lie below a. Note that if we sum up over all a, then $\sum_a r_a = |T|$ (every triangle is counted once, for its middle vertex), $\sum_a m_a = 2|T|$ (every triangle is counted twice), and $\sum_a n_a = \binom{n}{2}$.

As remarked before, a crossing in the graph G_a (between two segments, or between a ray and a segment, or between two rays) corresponds, in a one-one fashion, to a pinched crossing between triangles with common vertex a. Therefore, if we denote the number of crossings in G_a by x_a, then $x \geq \sum_a x_a$.

The second observation is that the graph G_a inherits the interleaving property from T. Thus, we can decompose the edges of G_a into convex chains. This time, a chain may end either at a vertex or in a ray. Therefore, the number c_a of chains equals $(n_a + r_a)/2$. Again, we use the convex chains to analyze the number of crossings in G_a, but this time to derive a lower bound. Consider a pair C, C' of such chains. For technical reasons that will become clear below, if one of the chains either starts or ends at a vertex, we call the pair uninteresting and do not analyze it further. There are at most $c_a n_a$ such pairs. For the remaining pairs, there are three possibilities: If C and C' intersect in a proper crossing between two edges, we are happy because we can reconstruct the pair of chains from the crossing pair of edges, so every crossing is counted at most once. It remains to bound the number of pairs that do cross in this nice fashion. The chains C and C' might intersect at a vertex v of G_a. In this case, if we know one edge of C incident to v, then we know both C and, up to two possibilities, v; in order to also encode C', it suffices to know an edge of C' incident to v, and any vertex of G_a has degree at most n. Therefore, there are at most $2nt_a$ pairs of chains that intersect at a vertex. Finally, if C and C' do not intersect at all, then, since we assume that both start and end in rays, one of them must lie completely above the other. In this case, we can encode the pair as follows: Fix one edge e of C. If we translate the line through e upwards, then it will first meet C' at a unique vertex v, and we can reconstruct C and C' if we know e and v. Therefore, there are at most $n_a t_a$ pairs of disjoint chains. Alltogether, we get $x_a \geq \Omega(c_a^2) - O(nt_a)$. If we sum up over all $a \in S$, then by substituting $c_a^2 = (r_a^2 - 2n_a r_a + n_a^2)/4$ and by applying Hölder's inequality $(\sum_a r_a^2 \geq (\sum_a r_a)^2/n = |T^2|/n)$, we arrive at the conclusion of Lemma 8.1.

8.2. Intersecting Simplices in Higher Dimensions. Suitable generalizations of the notion of crossing edges to triangles played a key role in the improvements in three dimensions. This raises the question, what the "right" generalization to higher dimensions would be. Dey and Pach [**55**] studied several problems of this kind for geometric hypergraphs (S,T) in $\mathbf{R}^d$, mostly in the case that are either d-regular (i.e., the simplices are $(d-1)$-dimensional) or $(d+1)$-regular. Simplices $\tau_1, \ldots, \tau_r$ are called *strongly crossing* if they are pairwise vertex-disjoint and their relative interiors have a common point of intersection. For $r > 2$, a weaker notion is that of *pairwise crossing* simplices, which means that any two simplices cross, but there need not be a point common to all of them. Among other things, Dey and Pach show the following:

Theorem 8.3. *Let (S,T) be a geometric hypergraph on n points in $\mathbf{R}^d$.*

(1) *If (S,T) is d-regular (i.e., the simplices in T are $(d-1)$-dimensional) and T contains no crossing pair of simplices, then $|T| \leq O(n^{d-1})$, and this bound is tight up to the constant factor.*
(2) *If (S,T) is $(d+1)$-regular and T contains no pair of crossing simplices, then $|T| = O(n^d)$, and this bound is tight up to the constant factor.*

They also derive upper bounds of $O(n^{d-(1/d)^{r-2}})$ for d-regular hypergraphs that do not contain an r-tuple of pairwise crossing simplices for $r > 2$, but this bound does not seem to be tight.

Note that the condition that there are no r mutually crossing full-dimensional simplices is closely related to the of Colorful Tverberg Theorem, which asserts the existence of r strongly crossing dimensional simplices in a geometric (or, more generally, a topological) complete r-partite $(d+1)$-uniform hypergraph $K^{(d+1)}(t, \ldots, t)$, $t = t(r,d)$, in $\mathbf{R}^d$.

8.3. Planes Crossing Halving Simplices in $\mathbf{R}^4$. Matoušek, Sharir, Smorodinsky, and Wagner [**101**] found a way to avoid the Selection Lemma also in the 4-dimensional case and obtained the following improved bound:

Theorem 8.4. *Let (S,T) be an interleaving 4-uniform geometric hypergraph in $\mathbf{R}^4$, with $t := |T|$ hyperedges and $n := |S|$ points. Then*

$$t = O(n^{4-2/45}).$$

The main new idea is to consider the intersection of the simplices in T with a 2-dimensional plane, instead of a line. Let us say that a 3-dimensional simplex σ and a 2-dimensional plane π in $\mathbf{R}^4$ *cross* if π does not intersect any of the edges of σ but meets the relative interior of τ in a line segment, whose endpoints are the intersections of π with two of the bounding triangles of σ. (This is the generic way a 2-plane and a 3-simplex intersect in $\mathbf{R}^4$, if at all.) In accordance with the general strategy, the proof of Theorem 8.4 consists of two steps:

LEMMA 8.5. *Let (S,T) be a 4-uniform geometric hypergraph in $\mathbf{R}^4$, with $n = |S|$ points and $t = |T|$ simplices. If $t > Cn^{11/3}$, for some absolute constant $C > 0$, then there is a 2-dimensional plane π that crosses $\Omega(t^3/n^8)$ simplices of T.*

This lemma can be derived from Theorem 8.3 by projecting the simplices in $\mathbf{R}^3$. In the projection, we get full-dimensional simplices, so if $t \geq \Omega(n^3)$, there are two that cross. It is not hard to show that for two crossing 3-dimensional simplices, there is always an edge of one that crosses the other. Applying random sampling, one can show that there must in fact be $\Omega(t^3/n^6)$ crossings between an edge and a 3-simplex, and so some edge crosses at least $\Omega(t^3/n^8)$ 3-simplices. Lifting the line through this edge back to $\mathbf{R}^4$, one gets the desired plane.

The more difficult part of the proof is to establish the following analogue of Lovász' Lemma for 2-dimensional planes instead of lines.

LEMMA 8.6. *Let (S,T) be an interleaving 4-uniform geometric hypergraph on n points in $\mathbf{R}^4$ in general position. Then no 2-dimensional plane crosses more than $O(n^{4-2/15})$ simplices of T.*

The idea is that the intersection of such a 2-plane π with the hypergraph (S,T) defines a geometric graph $G = (V,E)$ in π: An edge is the intersection of a 3-simplex $\tau \in T$ with π, and a vertex is the intersection of a π with a triangle abc, spanned by points in S. Moreover, it is not hard to see that the graph G inherits the interleaving property. However, the number of vertices of G can be $\Theta(n^3)$, so at first sight there might be $\Theta(n^6)$ crossings in G, hence a direct application of the Crossing Lemma only yields the trivial upper bound of $O(n^4)$ for the number of edges in E.

To overcome this difficulty, it becomes necessary to exploit more of the structure of G. Each edge of G connects two vertices labeled by triples abc and abd of points in S that share a common pair ab (the edge corresponds to the simplex $\tau = abcd \in T$). Thus, the possible edges are very restricted. The basic idea is to use a blow-up of the graph, by considering several auxiliary graphs whose edges correspond to short paths in G and by studying the drawings of these auxiliary graphs induced by the given geometric drawing of G. The details are somewhat complicated, and we refer the reader to [**101**].

9. Convex Quadrilaterals

Crossing numbers of graphs are a fundamental notion in discrete and computational geometry and also of considerable practical importance, for instance in VLSI design [**93**]. Determining the crossing number of a given graph is an NP-hard problem [**70**], and even for some very basic classes of graphs it is not known what their crossing number is. We refer to Brass, Moser and Pach [**35**, Chapter 9] for a survey of crossing numbers and related problems. As we have seen, crossing numbers are in particular an important tool for studying k-edges of planar point sets, but there are also interesting implications in the other direction.

Determining the crossing numbers of complete graphs and of complete bipartite graphs, respectively, are among the oldest problems in the area. In fact, the notion of crossing number was invented by Turán [**138**] when he posed the latter question, also known as "Turán's Brick Factory Problem". For complete graphs, Guy [**74**] conjectured that

$$\mathrm{cr}(K_n) \stackrel{(?)}{=} \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

This conjectured optimum is asympotictally equal to $\frac{3}{8}\binom{n}{4}$. There are constructions that achieve as few crossings [**76**]. An alternative construction, closer to the spirit of this survey, is the following:

Spherical geodesic drawings. Let U be a set of n unit vectors in general position on the sphere $\mathbf{S}^2$, and suppose that we connect any wo points of U by the shorter spherical geodesic arc between them. We call the resulting drawing of K_n a *geodesic drawing.*

Observation 9.1. *Let U be the vertex set of a geodesic drawing of K_n on $\mathbf{S}^2$, and let V be the Gale dual configuration of n vectors in $\mathbf{R}^{n-3}$. Then the crossings in the geodesic drawing are in one-to-one correspondence with the vector halving facets of V (i.e., with the $(n-4)$-element subsets $F \subset V$ such that the linear hyperplane spanned by F has two of the remaining vectors on either side).*

Corollary 9.2. *Consequently, if we take V to be a neighborly vector configuration, i.e., the homogenization of the vertex set of a neighborly $(n-4)$-polytope, then the number of crossings equals Guy's conjectured optimum above.*

The observation follows from the fact that there are only three different combinatorial types of configurations of four unit vectors in general position on $\mathbf{S}^2$, corresponding to the sign pattern of the unique (up to a nonzero scalar factor) linear dependence among the vectors: Either the four vectors contain the origin in their convex hull (*Type* 0, sign pattern $++++$ or $----$); or they can be separated from the origin by a plane. In this case, their radial projection onto that plane either looks like a triangle with a fourth point inside (*Type* 1, sign pattern $+++-$ or $---+$), or like a convex quadrilateral (*Type* 2, sign pattern $++--$). Only in the last case, for Type 2, do we get a crossing between the six geodesic arcs, namely exactly one, between the diagonals of the quadrilateral. It follows directly from Gale duality that for $Q \in \binom{[n]}{4}$, $\{u_i : i \in Q\}$ is of Type k iff $\{v_i : i \in [n] \setminus Q\}$ is a k-facet of V. The corollary then follows from Proposition 4.1.

Thus, the conjectured optimum can be achieved by a very simple kind of drawing. On the other hand, the best general lower bound to date is $\mathrm{cr}(K_n) \geq (3/10+o(1))\binom{n}{4}$, which follows from the work of Kleitman [**89**] who determined the crossing number of the complete bipartite graphs $K_{5,n}$ and $K_{6,n}$. It is natural to ask if we can prove better lower bounds for more restricted kinds of drawings.

9.1. Rectilinear Crossing Numbers. A *rectilinear drawing* of a graph is a drawing in the plane such that all the edges are straight-line segments (we do not require the edges to be axis-parallel, however, as the name might seem to suggest). Rectilinear drawings in $\mathbf{R}^2$ are a special case of spherical geodesic drawings on the sphere, with the additional requirement that all the vertices are contained in an open hemisphere (the equivalence is by radial projection onto a tangent plane). In this case, Type 0 is impossible and all quadrupels of vertices are of Type 1 or of Type 2.

The number of crossings in a rectilinear drawing of K_n on a point set S equals the number $\square(S)$ of convex quadrilaterals of S (i.e., of 4-element subsets of Type 2). Thus, determining the rectilinear crossing number $\overline{\text{cr}}(K_n)$ of complete graphs is equivalent to determining $\square(n) := \min_{|S|=n} \square(S)$, the minimum number of convex quadrilaterals in any n-point set in the plane.

The problem has received much attention in recent years with regard to both proving asymptotic bounds as well as to determining the exact value for small cases. Another motivation to study this problem is provided by the fact that the rectilinear crossing number of complete graphs determines the rectilinear crossing number of random graphs, see Spencer and Tóth [**129**].

Upper Bounds. Constructions of point sets with few convex quadrilaterals were given by Jensen [**80**], Singer [**127**], and others. The currently best upper bound is

$$\square(n) \leq 0.3807\binom{n}{4} + O(n^3), \tag{3}$$

due to Aichholzer, Aurenhammer and Krasser [**10**]. The best construction "by hand", i.e., without a computer-generated base case, yields an upper bound of $0.3838\binom{n}{4}$ and is due to Brodsky, Durocher, and Gethner [**38**].

Sylvester's Four-Point Problem. There is an equivalent version of the problem, the classical *Four-Point Problem* from geometric probability [**119**]. For a (Borel) probability ditribution μ in the plane, let $\square(\mu)$ denote the probability that four independent μ-random points form a convex quadrilateral. We assume that every line has μ-measure zero, so that degenerate configurations occur only with probability zero. If μ is the uniform distribution on the unit circle, then $\square(\mu) = 1$, so the interesting question is how small $\square(\mu)$ can be. This problem was first posed by Sylvester [**132**] in 1864 (without, however, rigorously addressing the issue of the dependence on the underlying distribution).

Initially, investigations focussed on the case of a uniform distribution μ_K on a convex body K in the plane. For this special case, the problem was solved by Blaschke [**33**], who showed that

$$\frac{2}{3} \leq \square(\mu_K) \leq 1 - \frac{35}{12\pi^2} \approx 0.704,$$

and that both inequalities are sharp: The lower bound is attained iff K is a triangle, and the upper bound iff K is an ellipse.

This, however, leaves the problem unresolved for general μ. Scheinerman and Wilf [**121**] pointed out that

$$\inf_{\mu} \square(\mu) = \lim_{n\to\infty} \frac{\square(n)}{\binom{n}{4}},$$

(a double-counting argument shows that the sequence on the right-hand side is monotonically increasing, so the limit exists.) Thus, Sylvester's Four-Point Problem for general distributions is equivalent to determining the exact asymptotics of $\square(n)$.

9.2. Convex Quadrilaterals and k-Edges. Lovász, Vesztergombi, Wagner, and Welzl [**96**] and independently Ábrego and Fernandez-Merchant [**1**] pointed out the following close connection between convex quadrilaterals and k-edges. For a set S of n points in general position in the plane, et $\triangle = \triangle(S)$ denote the number of 4-element subsets of Type 1 (triangles with an interior point). Then $\triangle(S) + \square(S) = \binom{n}{4}$. Moreover, we can write the right-hand side of the equation as $\binom{n}{4} = \frac{(n-3)(n-4)}{24} \sum_k e_k(S)$, because the latter sum equals $2\binom{n}{2}$ (every directed edge spanned by the point set is counted once).

To get another linear equation involving these quantities, consider the sum $\sum_{Q \in \binom{S}{4}} e_0(Q)$, i.e., the combined number of convex hull edges of all 4-point subsets. On the one hand, this equals $3\triangle(S) + 4\square(S)$. On the other hand, given a k-edge pq, we have $\binom{n-2-k}{2}$ ways of choosing two more points such that pq is a 0-edge of the resulting four points. Thus, $3\triangle(S) + 4\square(S) = \sum_k \binom{n-2-k}{2} e_k(S)$. Combining these two equations and simplifying (using symmetry $e_k = e_{n-2-k}$, one arrives at

Lemma 9.3. *For every set of n points in the plane in general position,*

$$\square(S) = \sum_{k < \frac{n-2}{2}} \left(\frac{n-2}{2} - k\right)^2 e_k(S) - \frac{3}{4}\binom{n}{3}.$$

Having expressed $\square$ (up to a lower order error term) as a positive linear combination of the e_k's, we can substitute any lower estimates for the numbers e_k to obtain a lower bound for $\square$. Using an exact version of Lovász' Lemma in the plane (see Section 10.1), it is not hard to derive a sharp lower bound for each individual e_k (see [**96**]):

Proposition 9.4. *For every set S of n points in the plane in general position and for every $0 \leq k < \frac{n-2}{2}$,*

$$e_k(S) \geq 2k + 3.$$

For every $j \geq 0$ and $n \geq 2j + 3$, this bound is attained.

Substituting this bound into Lemma 9.3, one only obtains a weak lower bound of $\square(n) \geq \frac{1}{4}\binom{n}{4} + O(n^3)$. Moreover, the bound in Proposition 9.4 is sharp: a point set for which this bound is attained consists of the vertices of a regular $(2k+3)$-gon plus $n - 2k - 3$ points very close to the center of the polygon.

However, this example is highly tuned to a particular value of k. To obtain a stronger lower bound for $\square(n)$, one uses summation by parts to rewrite the linear combination in Lemma 9.3 in terms of the numbers $e_{\leq k} = \sum_{j \leq k} e_j$.

Lemma 9.5. *For every set S of n points in the plane in general position,*

$$\square(S) = \sum_{k < \frac{n-2}{2}} (n - 2k - 3) e_{\leq k}(S) - \frac{3}{4}\binom{n}{3} + c_n,$$

where $c_n = \frac{1}{4} e_{\leq \frac{n-3}{2}}$ if n is odd, and $c_n = 0$ if n is even.

Note that in either case, the last two terms are $O(n^3)$.

9.3. Lower Bounds for $e_{\leq k}$. Lovász, Vesztergombi, Wagner, and Welzl [**96**] and Ábrego and Fernandez-Merchant [**1**] proved the following bound for the numbers $e_{\leq k}$ of at-most-k-edges: For any set S of n points in the plane in general position, and $k < \frac{n-2}{2}$,

$$e_{\leq k}(S) \geq 3\binom{k+2}{2}. \tag{4}$$

Together with Lemma 9.5, this yields a lower bound of $\square(n) \geq \frac{3}{8}\binom{n}{4} + O(n^3)$, which is asymptotically equal to Guy's conjecture for the crossing number of complete graphs.

The estimate in (4) is tight for $k < \lfloor n/3 \rfloor$ for "tripod-shaped" point sets $S = S_1 \cup S_2 \cup S_3$, $\lfloor n/3 \rfloor \leq |S_i| \leq \lceil n/3 \rceil$ for $i = 1, 2, 3$, such that each line spanned by two points in one part S_i strictly separates the other two parts from each other. However, in the middle range $\lfloor n/3 \rfloor \leq k < \frac{n-2}{2}$, improvements are possible. Observe, for instance, that for n odd and $k = \frac{n-5}{2}$, we have $e_{\leq k+1} = \binom{n}{2}$ and so $e_{\leq k} = \binom{n}{2} - e_{k+1} = 4\binom{k+2}{2} - o(k^2)$. Generally, for k very close to $n/2$, lower bounds for $e_{\leq k}$ are equivalent to upper bounds for the number of halving edges or to sums of the form $\sum_{k<j<\frac{n-2}{2}} e_j$. Based on this observation and using a result of Welzl [**144**], Lovász et al. derived a first improvement of $\square(n) \geq (3/8 + \varepsilon)\binom{n}{4}$ with $\varepsilon \approx 10^{-5}$. This is a small improvement, but it shows that that the crossing number and the rectilinear crossing number of complete graphs differ in the asymptotically dominating term. A more substantial improvement was obtained by Balogh and Salazar [**26**], and recently Aichholzer, Garcia, Orden, and Ramos [**12**] set the current record:

Theorem 9.6. *For any set of n points in general positition in the plane,*

$$e_{\leq k} \geq 3\binom{k+2}{2} + \sum_{j=\lfloor n/3 \rfloor}^{k} (3j - n + 3).$$

This bound is tight for $k < \lfloor 5n/12 \rfloor$ [**7**]. However, close to $n/2$, it still only yields a lower bound of roughly $(4 - \frac{1}{9})\binom{n}{4}$, so again improvements are possible. For $\square(n)$, Aichholzer et al. obtain a lower bound that comes quite close to the upper bound in Equation (3):

Corollary 9.7.

$$\square(n) > (41/108 + \varepsilon)\binom{n}{4} + O(n^3) > 0.379631\binom{n}{4}.$$

The starting point of Aichholzer et al. is the study structural properties of extremal examples:

Proposition 9.8. *Let S be a set of n points in general position in the plane.*

(1) *If S minimizes the number of convex quadrilaterals, then $e_0(S) = 3$, i.e., the convex hull of S is a triangle.*[14]
(2) *If $e_0(S) > 3$, then there exists another set S' of n points with $e_{\leq k}(S') \leq e_{\leq k}(S)$ for all $k \leq \lfloor \frac{n-2}{2} \rfloor$, and at least one of the inequalities is strict.*

The proof is by continuous motion. They key observation is the following: Let v be an extreme point of S and let r be a generic halving ray of S emanating from v to infinity (there are at least $\lfloor \frac{n-1}{2} \rfloor$ points of S on either side of the line spanned by r, and the r does not pass through the convex hull of S). Then, as we move

[14] This fact was independently proved by Balogh, Leaños, Pan, Richter, and Salazar [**25**]; in fact, their proof, as well as the proof of Lovász et al., Ábrego and Fernandez-Merchant, and Balogh and Salazar, work in the dual setting of pseudoline arrangements, or uniform oriented matroids of rank 3; the arguments of Aichholzer et al. can also be adapted to this slightly more general setting.

v from its original position along r, the only mutations that occur are of the form that a k-edge incident to v becomes a $(k+1)$-edge, for some $k \leq n/2-2$.

The second part of the above proposition also leads to a very simple proof of the estimate in (4), by induction on k: Let S be a set of n points minimizing $e_{\leq k}$. By the proposition, we may assume that $e_0(S) = 3$. Removing the three extreme points of S, we obtain a set S' of $n-3$ points. By induction, $e_{\leq k-2}(S) \geq \binom{k}{2}$ (here we use that $k < \frac{n-2}{2}$ implies $k-2 < \frac{n-5}{2}$), and each j-edge of S' is a $(j+1)$-edge or a $(j+2)$-edge of S. In the worst case, they are all $(j+2)$-edges of S. Moroever, the three extreme points contribute exactly three 0-edges of S and six j-edges of S, $1 \leq j \leq k$. Summing up, $e_k(S) \geq 3\binom{k}{2} + 3 + 6k = 3\binom{k+2}{2}$.

The improvement in the middle range is based on tightening the pessimistic estimate that every j-edge of S' is a $(j+2)$-edge of S. More precisely, Aichholzer et al. show that for $n/3 - 1 \leq j \leq (n-5)/2$, there are at least $3j - n + 3$ "good" j-edges of S' that are $(j+1)$-edges of S. This implies the bound in Theorem 9.6.

9.4. Small Cases. Tables 3 and 4 summarize what is known about $\mathrm{cr}(K_n)$, $\square(n) = \overline{\mathrm{cr}}(K_n)$, and $e^{(2)}_{1/2}(n)$ for small values of n (where we interpret $e^{(2)}_{1/2}(n)$ as the maximum number of (undirected) halving edges for even n and as $e^{(2)}_{\lfloor \frac{n-2}{2} \rfloor}(n)$ for odd n). The values of $\mathrm{cr}(K_n)$ for $n \leq 10$ and of $\square(n)$ for $n \leq 9$ were determined by Guy [**75**]. We remark that if Guy's conjecture is true for odd n, then a straightforward double-counting argument implies the conjecture also for the next even n. Recently, Pan and Richter [**116**] determined $\mathrm{cr}(K_{11})$ and hence also $\mathrm{cr}(K_{12})$. The values $\square(n)$ for $n \leq 9$ were also found by Guy; $\square(10)$ was determined by Brodsky, Durocher, and Gethner [**37**], and independently by Aichholzer, Aurenhammer, and Krasser [**9**]. The values of $\square(n)$ for $13 \leq n \leq 17$, are taken from Aichholzer and Krasser [**11**]. The numbers $e^{(2)}_{1/2}(n)$, for even $n \leq 12$ were determined by Aronov, Andrzejak, Har-Peled, Seidel, and Welzl [**19**] using their Crossing Identity for halving edges. The value of $e^{(2)}_{1/2}(14)$ and the upper bound for $e^{(2)}_{1/2}(16) \leq 28$ were established by Beygelzimer and Radziszowski [**30**]; the lower bound $e^{(2)}_{1/2}(16) \geq 27$ is due to Eppstein [**62**]. Using Theorem 9.6, it is not hard to fill in the remaining values of $e^{(2)}_{1/2}(n)$ for odd $n \leq 11$. All remaining entries are taken from Aichholzer, Garcia, Orden, and Ramos [**12**].

For an up-to-date database of small values of $\square(n)$, including coordinates for extremal point configuration and a study of the number of extremal examples, we refer the reader to Aichholzer's webpage [**8**].

n	4	5	6	7	8	9	10	11	12
$\mathrm{cr}(K_n)$	0	1	3	9	18	36	60	100	150

TABLE 3. The crossing number of small complete graphs.

10. Connections to the Combinatorial Theory of Convex Polytopes

Levels in arrangements and the polar dual notions of k-facets and k-sets of point sets (or more generally, (i,j)-partitions) are generalizations of faces of convex polyhedra, given as the intersection of finitely many halfspaces, or in the dual as convex hulls of point sets.

n	4	5	6	7	8	9	10	11	12	13	14	15
$\square(n)$	0	1	3	9	19	36	62	102	153	229	324	447
$e^{(2)}_{1/2}(n)$	3	7	6	12	9	18	13	24	18	31	22	39

n	16	17	18	19	20	21	22	23	24
$\square(n)$	603	798	1026 1029	1318	1652 1657	2055	2521 2528	3075 3077	3690 3699
$e^{(2)}_{1/2}(n)$	27 28	47	33 36	56	38 43	66	44 51	75 76	51 60

TABLE 4. $\square(n)$ and $e_{1/2}(n)$ for small n.

The combinatorial theory of convex polyhedra and their face numbers is highly developed, including exact upper bounds in all dimensions (McMullen's *Upper Bound Theorem* [**103**]) and, in the case of general position, even a complete characterization of the possible vectors of face numbers (the *g-Theorem* proved by Stanley [**131**] and Billera and Lee [**31**]).

Some of these results have interesting reformulations in terms of k-facets and levels, mostly based on Gale duality and its relatives, and in a few cases, results concerning polytopes have been (partially) extended to higher single levels or $(\leq k)$-levels. In this section, we discuss some of these connections and generalizations.

10.1. An Exact Version of Lovász' Lemma. We begin in the setting of Lovász' Lemma. Our presentation follows Welzl [**145**]. Closely related are the articles of Lee [**92**], Clarkson [**48**], and Mulmuley [**107**]. Given a line ℓ and a $(d-1)$-dimensional simplex σ in $\mathbf{R}^d$, we say that ℓ *crosses* σ if ℓ intersects the relative interior of σ in a single point. It will be convenient to assume that ℓ is directed and that σ is cooriented. Under this assumption and if ℓ crosses σ, then we say that it *enters* σ if it is directed from the positive to the negative side of σ; otherwise, we say that ℓ *leaves* σ. We use the same terminology when talking about directed segments or semiinfinite rays instead of directed lines.

10.1.1. *The h-vector of a point set and a line.* Consider a set S of n points and a directed line ℓ in $\mathbf{R}^d$. We assume that S is in general position, and that ℓ is in general position with respect to S, i.e., that ℓ does not intersect the convex hull of any $d-1$ or fewer points of S.

For integer k, we define $h_k = h_k(S,\ell)$ as the number of k-facets of S that are entered by ℓ, and we call $(h_0(S,\ell),\ldots,h_{n-d}(S,\ell))$ the *h-vector* of S and ℓ.

The first observation is that this h-vector depends only on the orthogonal projection onto the hyperplane $\ell^\perp \cong \mathbf{R}^{d-1}$ orthogonal to ℓ. Let $\overline{S}$ be the image of S under this projection, and let o be the point onto which ℓ projects. Note that by our assumptions, $\overline{S} \cup \{o\}$ is in general position. For integer r, let $f^r = f^r(\overline{S},o)$ denote the number of $(d+r)$-element subsets of $\overline{S}$ whose convex hull contains o. Equivalently, $f^r(\overline{S},o)$ equals the number of $(d+r)$-element subsets $R \subseteq S$ whose convex hull is intersected by ℓ. Given such a subset, there is a unique "topmost" facet of $\mathrm{conv}(R)$ that is intersected by ℓ (where we think of ℓ as defining the "vertical direction"). By coorienting this facet "downwards", we obtain an k-facet σ of S, for some $k \geq r$, that is entered by ℓ.

Conversely, given a k-facet σ of S that is entered by ℓ, there are $\binom{k}{r}$ ways to choose additional r points in order to obtain a $(d+r)$-element subset R whose convex hull is intersected by ℓ and with topmost facet σ. Thus,

$$f^r(\overline{S}, o) = \sum_k \binom{k}{r} h_k(S, \ell) \tag{5}$$

for all r. This system of linear equations is invertible, i.e., the vectors $(h_0, \ldots, h_{n-d})$ and $(f^0, \ldots, f^{n-d})$ determine each other. Consequently, the h-vector of S and ℓ only depends on $\overline{S}$ and o and is independent of the "lifting" to S and ℓ. In particular, the h-vector does not change if we translate the points of S parallel to ℓ or if we reverse the orientation of ℓ. The latter observation implies the so-called *Dehn-Sommerville Equations*,

$$h_k(S, \ell) = h_{n-d-k}(S, \ell) \tag{6}$$

for all k.

Next, we will se how the numbers $f^r(\overline{S}, o)$ can also be expressed directly within $\mathbf{R}^{d-1}$, without the detour through the lifting to $\mathbf{R}^d$.

10.1.2. *Winding numbers.* Let X be a set of n points in general position in $\mathbf{R}^d$. As mentioned above (Remark 7.10), the k-facets of X form a closed, (co)oriented "hypersurface"[15]. As a consequence, for any point $o \in \mathbf{R}^d$ not lying on this hypersurface, we can define an integer *winding number* g_k of the k-facet hypersurface around o. These winding numbers were first considered by Lee [**92**]. Our presentation follows [**145**]. Specifically, let $o \in \mathbf{R}^d$ be such that $X \cup \{o\}$ is in general position. Choose a semiinfinite ray ρ from infinity to o that does not intersect any $(d-2)$-dimensional simplex spanned by X. As in the case of directed lines, we say that the ray ρ enters a k-facet σ if it intersects the relative interior of σ in a single point and is directed from the positive to the negative side of σ. In the case of the opposite orientation, we say that ρ *leaves* σ. Then $g_k(X, o)$ is defined as the number of k-facets entered by ρ minus the number of k-facets left by ρ. By extending ρ into a directed line that continues beyond o (or conversely, by clipping such a line), we see that

$$g_k(X, o) \leq h_k(X, \lambda) \tag{7}$$

for any directed line λ through o. If o has depth at least $k+1$, then we have equality (since we cannot leave any k-facets), but this is not a necessary condition. The fact that $g_k(S, o)$ does not depend on the choice of ρ follows from the properties of the k-facet "hypersurface", but also directly from the following lemma:

LEMMA 10.1. *For any set X of n points in $\mathbf{R}^d$,*

$$f^r(X, o) = \sum_k -g_k(X, o) \binom{k}{r+1}.$$

The proof of this lemma is based on a double-counting argument similar to that for Equation (5) above, see [**145**].

As pointed out by Lee, a deep theorem about convex polytopes, the so-called *Generalized Lower Bound Theorem*, implies that the winding numbers g_k are nonnegative for $k \leq (n-d-1)/2$; in fact, by Gale duality, this statement is equivalent to the GLBT. We will see a version of this equivalence below. For $k <$

[15] More precisely, a union of (co)oriented simplicial pseudomanifolds; the decomposition can be obtained by rotations.

$(n-d-d^2)/(d+1)$, this nonnegativity follows easily from the existence of centerpoints. For larger k, no elementary proof is known, except in dimension $d \leq 3$. For $d = 3$, Sharir and Welzl [**126**] showed that the nonnegativity of the winding numbers g_k, $k \leq (n-4)/2$, is equivalent to the following result of Pach and Pinchasi [**113**]:

THEOREM 10.2. *Let B ("black") and W ("white") be two disjoint planar point sets of n elements each such that $B \cup W$ is in general position. A line ℓ is called* balanced *if it is spanned by a black point and a white point and on both sides of ℓ, the number of black points minus the number of white points is the same. Then the number of of balanced lines is at least n^2.*

We return to the setting of Lovász' Lemma. When we apply the previous lemma to the projected set $X = \overline{S}$ in $\mathbf{R}^{d-1}$ and use (5) and summation by parts, we obtain the following: For any n-point set S and any directed line ℓ in general position in $\mathbf{R}^d$,

$$g_k(\overline{S}, o) = h_k(S, \ell) - h_{k-1}(S, \ell), \tag{8}$$

for all k. (In particular, $g_k = -g_{n-d+1-k}$, by (8).) Thus, $h_k(S,\ell) = \sum_{j \leq k} g_j(\overline{S}, o)$. By induction on the dimension and (7), we conclude:

THEOREM 10.3 (Exact Version of Lovász' Lemma). *For a set S of n points and a directed line ℓ in general position in $\mathbf{R}^d$,*

$$h_k(S, \ell) \leq \binom{k+d-1}{d-1}$$

for all k. Moreover, this upper bound is attained iff in the projection onto $\ell^\perp \cong \mathbf{R}^{d-1}$, the point o has depth at least $k+1$ with respect to $\overline{S}$.

COROLLARY 10.4. *Let $X \subset \mathbf{R}^d$, $|X| = n$, and $o \in \mathbf{R}^d \setminus X$. Then*

$$f^r(X, o) \leq \sum_{j=0}^{\lfloor \frac{n-d-1}{2} \rfloor} \binom{j}{k}\binom{j+d}{d} + \sum_{j=0}^{\lceil \frac{n-d-2}{2} \rceil} \binom{n-d-1-j}{k}\binom{j+d}{d}.$$

Equality is attained iff X is perfectly centered around X.

This corollary and the preceeding theorem are equivalent, by Gale duality, to McMullen's Upper Bound Theorem for convex polytopes, which we will discuss in the following section. We remark that analogues of these results have also been obtained in a continuous setting, where point sets are replaced by continuous probability distributions [**143**].

10.2. The Upper Bound Theorem and Some Generalizations. A convex d-dimensional polytope P can be described as the convex hull of n points in $\mathbf{R}^d$. McMullen's Upper Bound Theorem [**103**] (UBT) gives exact upper bounds for the complexity of the boundary of P in terms of n and d, together with a characterization of the extreme cases:

THEOREM 10.5 (Upper Bound Theorem). *Let S be a set of n points in $\mathbf{R}^d$. Then*

$$e_0(S) \leq e_0(C_{n,d}),$$

where $C_{n,d}$ is any set of n points on the moment curve $\gamma = \{(t, t^2, t^3, \ldots, t^d) : t \in \mathbf{R}\}$ *(the convex hull of $C_{n,d}$ is called a* cyclic polytope*). Moreover, equality is attained iff S is neighborly.*

Under polar duality, the convex hull of n points in $\mathbf{R}^d$ corresponds to the intersection of n hemispheres in $\mathbf{S}^d$, or of n affine halfspaces in $\mathbf{R}^d$. The dual form of the UBT reads as follows:

THEOREM 10.6 (Polar Dual UBT). *Let $\mathcal{A}$ be an arrangement of n hemispheres in $\mathbf{S}^d$. Then*

$$v_0(\mathcal{A}) \leq v_0(\mathcal{C}^*_{n,d})$$

where $\mathcal{C}^*_{n,d}$ is a polar-to-cyclic arrangement *of hemispheres. Moreover, equality is attained iff $\mathcal{A}$ is* polar-to-neighborly*, i.e., iff the intersection of any $\lfloor d/2 \rfloor$ bounding great $(d-1)$-spheres contains a vertex at level* 0*.*

The Upper Bound Theorem also gives exact upper bounds for the numbers of faces of intermediate dimensions $0 < r < d-1$, but here we will mostly focus on facets and vertices. We return to this issue later on.

We remark that the notion of a polar-to-cyclic arrangement without further specifications is only well-defined for spherical arrangments; in the affine case, the arrangement depends on the choice of the additional "northern hemisphere". For the 0-level, however, the exact choice is immaterial, as long as the "equator", i.e., the boundary of the northern hemisphere, does not intersect the 0-level, which we can always achieve.

The Upper Bound Theorem has been generalized in numerous ways. Most of these generalizations take the primal version, Theorem 10.5, as their starting point, and show that the same upper bound holds for the number of faces of more general $(d-1)$-dimensional simplicial complexes on n vertices: For simplicial $(d-1)$-spheres [**130**], for Eulerian complexes (provided n is sufficiently large) [**88**], and for several classes of simplicial manifolds and pseudomanifolds [**110, 78, 111**]. Another far-reaching extension is Kalai's [**82**] so-called *Strong Upper Bound Theorem* concerning subcomplexes of the boundary complex of a simplicial polytope.

Most of these extensions have a topological or algebraic flavor. Eckhoff [**57**], Linhart [**94**], and Welzl [**145**], independently of one another proposed the following, more geometric generalization concerning $(\leq \ell)$-levels, which we refer to as the *Spherical Generalized Upper Bound Conjecture* (*SGUBC*).

CONJECTURE 10.7 (SGUBC). *Let $\mathcal{A}$ be an arrangement of n great hemispheres in $\mathbf{S}^d$. Then*

$$v_{\leq \ell}(\mathcal{A}) \leq v_{\leq \ell}(\mathcal{C}^*_{n,d})$$

for $0 \leq \ell \leq (n-d-1)/2$. Equality holds iff $\mathcal{A}$ is polar-to-neighborly.

Eckhoff and Welzl formulated this conjecture in the feasible case and the dual setting of $(\leq \ell)$-facets of point sets. The SGUBC is known to be true in dimension $d = 2$, as shown by Peck [**117**] and by Alon and Győry [**16**]; and in dimension 3 provided the intersection of the hemispheres is nonempty [**145**]. The explicit upper bounds are $n(\ell+1)$ and $2\left(\binom{\ell+2}{2}n - 2\binom{\ell+3}{3}\right)$ for dimensions 2 and 3, respectively.

As observed in [**145**], for arrangements of hemispheres, the restriction $\ell \leq (n-d-1)/2$ is crucial if we are striving for exact bounds: For instance, in the case $d = 2$ and $k = 0$, let v_ℓ denote the number of vertices at level ℓ. In the spherical case, $v_\ell = v_{n-2-\ell}$. Thus, if n is even, then $2v_{\leq(n-2)/2} = 2\binom{n}{2} + v_{(n-2)/2}$. Consequently, the bound $v_{\leq\ell} \leq n(\ell+1)$ does not hold for $\ell = (n-2)/2$, else we would get an linear upper bound of $v_{(n-2)/2} \leq n$ for the middle level, contradicting the known superlinear lower bounds mentioned above.

The SGUBC is also related to the so-called Generalized Lower Bound Theorem (which is part of a complete combinatorial characterization of the face numbers of simplicial convex polytopes, the g-Theorem, conjectured by McMullen and proved by Stanley [**131**]and by Billera and Lee [**31**]). For instance, Welzl [**145**] showed that the SGUBC for arrangements in $\mathbf{S}^3$ is equivalent, by Gale duality, to the GLBT for d-polytopes with $d+4$ vertices.

Recently, the following weaker bound was proved [**142**]:

THEOREM 10.8 (2AGUBT). *Let $\mathcal{A}$ be an arrangement of n affine halfspaces in $\mathbf{R}^d$. Then*

$$v_{\leq\ell}(\mathcal{A}) \leq 2 \cdot v_{\leq\ell}(\mathcal{C}^*_{n,d})$$

for $0 \leq \ell \leq n-d$ (where $\mathcal{C}^*_{n,d}$ denotes a polar-to-cyclic spherical *arrangement as in the SGUBC).*

A sharp version of Theorem 10.8, without the factor of 2, was conjectured by Linhart, who proved it for $d \leq 4$; we refer to this sharp version as the *Affine Generalized Upper Bound Conjecture* (*AGUBC*); it is sharp for $\ell < \lceil n/(d+1)\rceil$, because it is equivalent to the SGUBC in that range. (This follows by taking the polar dual of a cyclic polytope with the origin at a center point of the vertex set). Moreover, the AGUBC, if it is true, might even be sharp for all $\ell \leq (n-d)/2$. This is the case if there are neighborly d-polytopes with n vertices whose vertex set are perfectly centered in the sense of Observation 2.6. Such perfectly centered neighborly poytopes are easily seen to exist in dimensions 2 and 3, but for higher dimensions, this remains open.

10.3. *h*-Vectors and *h*-Matrices. One of the central notions in the proof of the Upper Bound Theorem and its topological generalizations is a certain linear transformation of the vector of faces, the *h-vector*. For a convex polytope given as the intersection of halfspaces in general position, the h-vector has a simple geometric definition: Choose a linear functional φ that is not constant on any of the edges of the polytope, and orient every edge of the polytope in the direction of increasing φ. Then the j-th entry h_j is defined as the number of vertices of out-degree j, $0 \leq j \leq d$. If the polyhedron is bounded, these numbers turn out to be independent of the choice of φ.

Mulmuley [**107**] suggested a generalization of the h-vector to higher levels in arrangements. Let $\mathcal{A}$ be an arrangement of n affine halfspaces in $\mathbf{R}^d$. We refer to the affine subspaces that arise as intersections of bounding hyperplanes as *flats* of the arrangement; of particular interest to us will be the *lines*, i.e., the 1-dimensional flats.

Let φ be a linear functional $\mathbf{R}^d \to \mathbf{R}$. We call the pair $(\mathcal{A}, \varphi)$ a *linear program* (*LP*).[16] We assume that the LP is in *general position*, i.e., that any d of the bounding hyperplanes of the halfspaces intersect in exactly one point, no $d+1$ of the hyperplanes pass through a common point, and φ is not constant on any line of

[16] More precisely, the linear program is the optimization problem of maximizing the *linear objective function* φ over the *feasible region* $\bigcap \mathcal{A}$.

the arrangement. Under this assumption, once we fix a labelling $\{H_1, \ldots, H_n\}$ of the halfspaces in $\mathcal{A}$, the vertices of the arrangement are in 1-1 correspondence with the d-element subsets of $\mathcal{A}$, or with the d-element subsets $B \subset [n]$, called *bases*.

10.3.1. *Outdegrees and the h-matrix of an LP.* Let $v = \bigcap_{b \in B} \partial H_b$ be a vertex with basis B. If we drop an "equality constraint" (hyperplane) $a \in B$, we get a line $\ell = \bigcap_{b \in B \setminus \{a\}} \partial H_b$, and on this line two antipodal edges (possibly semiinfinite rays) incident to the vertex v. We consider the line ℓ to be oriented in the direction of increasing φ, hence one of the two antipodal edges is oriented towards v and the other away from v. Consider the edge that lies inside the halfspace H_a. If this distinguished edge is directed towards v, we call it *incoming*. otherwise it is *outgoing*, see Figure 30. (As far as v is concerned, we disregard the other edge that does not lie in H_a.) In other words, we get an ingoing edge if we "bump into" or "are stopped by" the halfspace H_a as we walk along ℓ in the direction of increasing φ, and otherwise we get an outgoing edge. The *outdgree* of v is the number of outgoing edges.

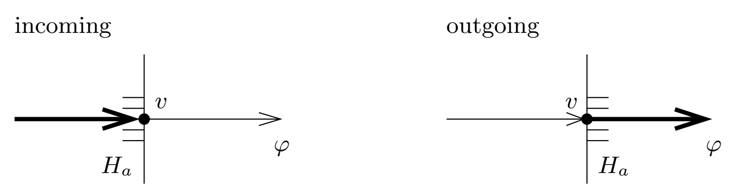


FIGURE 30. Incoming and outgoing edges.

DEFINITION 10.9 (h-Matrix). For $0 \leq j \leq d$ and $0 \leq \ell \leq n - d$, we define $h_{j,\ell}(\mathcal{A}, \varphi)$ as the number of vertices of out-degree j and level ℓ. If the LP $(\mathcal{A}, \varphi)$ is understood from the context, we simply write $h_{j,\ell}$.

The h-vector of the convex polyhedron $P = \bigcap \mathcal{A}$ is simply the zeroth column of the h-matrix. If P is a convex *polytope*, i.e., if it is bounded, then the h-vector is independent of the choice of the linear objective function φ. Mulmuley [**107**] showed that the same is true for the numbers $h_{j,\ell}(\mathcal{A}, \varphi)$, $0 \leq j \leq d$, provided the ℓ-level of the arrangement is bounded, see Corollary 10.14 below. In general, however, the entries of the h-matrix depend on φ, see Figure 31.

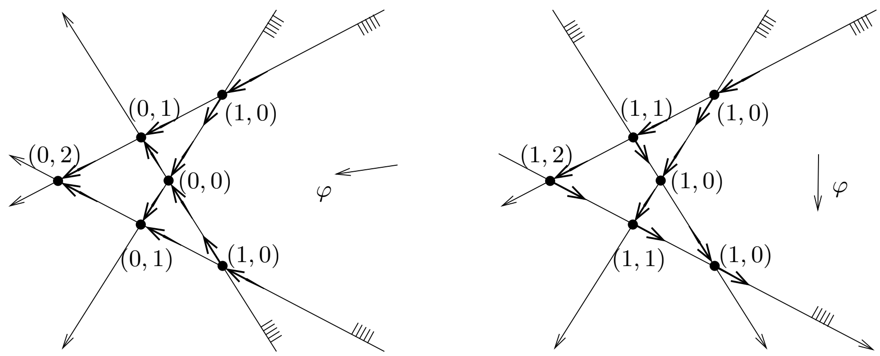
10.3.2. *LP-duality and the h-matrix.* To a d-dimensional LP $(\mathcal{A}, \varphi)$ with n halfspaces ("constraints"), there corresponds a *dual linear program* $(\mathcal{A}^*, \varphi^*)$ with n constraints in dimension $n - d$ (see, for instance, Gärtner and Matoušek [**71**]). (Our notation is somewhat misleading because both $\mathcal{A}^*$ and φ^* depend on both $\mathcal{A}$ and φ.) The halfspaces of the dual program are labeled by the same set $[n]$ of indices.

LP duality is closely related to Gale duality (Section 2.4), see [**32**]: We can write the primal program of maximizing φ over the polyhedron $P = \bigcap \mathcal{A}$ in (homogeneous) coordinates as

$$\max\{\langle a_{n+1}, x\rangle : x = (x_0, \ldots, x_d) \in \mathbf{R}^{d+1}, \langle a_i, x\rangle \geq 0 \text{ for } 1 \leq i \leq n, \langle a_0, x\rangle = 1\}$$

for some $(n+2)$ vectors $a_0, \ldots, a_{n+1} \in \mathbf{R}^{d+1}$, where $\langle , \rangle$ denotes the standard inner product. If $b_0, \ldots, b_{n+1} \in \mathbf{R}^{n-d+1}$ is the Gale dual vector configuration, then the dual program is given by

$$\max\{\langle b_0, y\rangle : y = (y_0, \ldots, y_{n-d}) \in \mathbf{R}^{d+1}, \langle b_i, y\rangle \geq 0 \text{ for } 1 \leq i \leq n, \langle b_{n+1}, x\rangle = 1\}.$$



$$[h_{j,\ell}] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [h_{j,\ell}] = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

FIGURE 31. In general, the h-matrix $[h_{j,\ell}]$ depends on the objective function φ. (In the figures, a label of (j, ℓ) indicates a vertex of out-degree j and level ℓ.)

Note that the roles of the "special indices" 0 and $n+1$ (homogenization and objective function) are exchanged. The dual of the dual is the primal.

The dual of a generic LP in general position is again in general position, and passing to complimentary index sets gives a one-to-one correspondence between the bases of the primal and bases of the dual program, $B \leftrightarrow B^* := [n] \setminus B$. Under this correspondence, it is not hard to verify the following properties:

LEMMA 10.10. *Let $(\mathcal{A}^*, \varphi^*)$ be the dual of $(\mathcal{A}, \varphi)$.*

(1) *For $B \in \binom{[n]}{d}$, $\varphi(B) = -\varphi^*(B^*)$ (the minus sign is only there because we want to have both the primal and the dual to be problems of maximizing a linear function).*
(2) *Conflicts and out-labels are dual to each other: If v is a primal vertex with basis B and v^* the dual vertex with basis $B^* = [n] \setminus B$, then an index $a \in B$ is the label of an outgoing edge at v iff in the dual, a is the index of a conflict, i.e., of a halfspace not containing v^*, and vice versa.*

COROLLARY 10.11. *LP duality transposes the h-matrix: For $0 \le j \le d$ and $0 \le \ell \le n-d$,*

$$h_{j,\ell}(\mathcal{A}, \varphi) = h_{\ell,j}((\mathcal{A}, \varphi)^*).$$

10.3.3. *Bounds for the h-matrix.* Because of LP duality and because of the difficulty of determining the complexity of a single level in a fixed dimension (respectively, because the latter is too large if we allow infeasible linear programs), we cannot expect to prove exact upper bounds for a single entry of the h-matrix or for the sum of entries in a single row or column. It turns out that the "right quantities" to bound are sums of entries in "upper left corners" of the h-matrix, see [**142**]:

Theorem 10.12. *For every generic d-dimensional linear program $(\mathcal{A},\varphi)$ with n constraints,*

$$h_{\leq j,\leq \ell}(\mathcal{A},\varphi) \leq 2\cdot \sum_{i=1}^{j} \binom{n-d-\ell+j}{i}\binom{d-j+\ell}{d-i}.$$

To derive Theorem 10.8 from this, choose any generic φ and get $v_{\leq \ell}(\mathcal{A}) = h_{\leq \lfloor d/2\rfloor,\leq \ell}(\mathcal{A},\varphi) + h_{\leq \lfloor \frac{d-1}{2}\rfloor,\leq \ell}(\mathcal{A},-\varphi)$. Substituting the bounds for the h-matrix and straightforward calculations lead to the desired bound.

Theorem 10.12 is proved by a reduction to a combinatorial lemma concerning quadrupels of finite sets with certain intersection and cardinality restrictions, which in turn is proved using tools from multilinear algebra. The factor of 2, which propagates to Theorem 10.8, seems to be an artifact of the current proof. Without this factor, the combinatorial lemma is tight for all values of the parameters, and Theorem 10.12 is tight for $\ell \leq n/d$ (and conceivably all the way up to $\ell \leq (n-d-1)/2$).

In the case $\ell = 0$ (or, symmetrically, $j = 0$), the combinatorial lemma holds true without the factor of 2 (and is known as the skew version of Bollobás's Theorem, see [**69**]), which implies

$$h_{\leq j,0} \leq \binom{n-d+j}{j}.$$

This was first pointed out by Alon and Kalai [**17**], who used this to give a simple proof of the original Upper Bound Theorem in the context of *collapsible complexes.* For polytopes, i.e., for the case of a bounded 0-level, McMullen proved stronger bounds for individual entries of the h-vector, $h_{j,0} \leq \binom{n-d+j-1}{j-1}$; this is also sometimes referred to as the UBT. Note that in the unbounded case, these exact bounds for individual entries of the h-vector are no longer correct. For a very simple counterexample see the arrangement on the right-hand side of Figure 31.

10.3.4. *Higher-dimensional faces.* Given a linear program $(\mathcal{A},\varphi)$, let f_k^r denote the number of r-dimensional faces of the arrangement $\mathcal{A}$ at level ℓ, let $\vec{f}_\ell^r$ denote the number of those faces that are bounded in the direction of φ (i.e., that have a finite φ-maximum, which will be attained at a vertex), and let $\bar{f}_\ell^r$ denote the number of those faces that are bounded both in the direction of φ and of $-\varphi$ (by genericity, these are precisely the faces that are bounded in the usual sense). A double-counting argument (charging every face to its "sink", i.e., its φ-maximal vertex, see [**107**]) yields

$$\vec{f}_\ell^r = \sum_{j=0}^{d}\sum_{s=0}^{r}\binom{j}{r-s}\binom{d-j}{s} h_{j,\ell-s}. \tag{9}$$

Together with Theorem 10.12, this implies

$$\bar{f}_{\leq \ell}^r \leq 2\cdot f_{\leq \ell}^r(\mathcal{C}_{n,d}^*).$$

Since the unbounded faces correspond to faces of a $(d-1)$-dimensional arrangement (in the "hyperplane at infinity"), induction on the dimension yields:

Corollary 10.13. $f_{\leq \ell}^r(\mathcal{A}) \leq 2\cdot \sum_{i=0}^{d} f_{\leq \ell}^{r-i}(\mathcal{C}_{n,d-i}^*)$.

The system of linear equations (9) is invertible (see [**107**]), and the columns $0,\ldots,\ell$ of the h-matrix depend only on the numbers $\vec{f}_k^r$, $0 \leq k \leq \ell$ and $0 \leq r \leq d$.

In particular, if all faces of the arrangement $\mathcal{A}$ at level $\leq \ell$ are bounded, then $\vec{f}_k^r = f_k^r$ is independent of φ, $0 \leq k \leq \ell$ and $0 \leq r \leq d$, and so are the rows $0, \ldots, \ell$ of the h-matrix.

Corollary 10.14 (Dehn-Sommerville Relations). *If all faces of $\mathcal{A}$ at level $\leq \ell$ are bounded, then the columns $0, \ldots, \ell$ of the h-matrix are independent of φ, and*

$$h_{j,k} = h_{d-j,k}$$

for $0 \leq k \leq \ell$ (as we see by reversing the direction of φ).

10.3.5. *Local maxima.* The zeroth row of the h-matrix of a linear program $(\mathcal{A}, \varphi)$ counts the number of vertices of out-degree 0, i.e., the *local maxima*, at level $k = 0, \ldots, n-d$. Under LP-duality, this corresponds to the zeroth column of the h-matrix of the dual LP $(\mathcal{A}^*, \varphi^*)$, i.e., to the h-vector of the polyhedron $\bigcap \mathcal{A}^*$. An LP in general position is bounded iff its dual is feasible. Thus, McMullen's bounds $h_{j,0} \leq \binom{n-d-+j-1}{j-1}$ for a d-dimensional bounded LP are equivalent to the fact that the number of local maxima at level k in a feasible d-dimensional LP is at most $\binom{d+k-1}{d-1}$. Clarkson [**48**] gave a different, direct proof for this using a random sampling approach.

Moreover, through polar duality, the local maxima at level k in a *feasible* arrangement correspond to the k-facets entered by a line in the setting of Section 10.1: Fix a labelling $\mathcal{A} = \{H_1, \ldots, H_n\}$ of the halfspaces and chose a coordinate system such that the linear objective function φ points vertically upwards (in the x_d-direction). Let $S = \{p_1, \ldots, p_n\} \subset \mathbf{R}^d$ be the polar dual set of points under vertical point-hyperplane duality, and let ℓ be the (directed) x_d-axis. Then a vertex v at level k in $\mathcal{A}$ corresponds to a lower k-facet of S. Moreover, the ordering of the vertices of $\mathcal{A}$ corresponds to the ordering of the dual $*$-facets according to the intersection of ℓ with the hyperplane spanned by the facet.

Dropping one of the equality constraints (bounding hyperplanes) from v and moving into he corresponding halfspace H_a corresponds to dropping the corresponding point a of σ and rotating upwards about the complementary ridge, so that a moves into the lower halfspace. Thus, a is the label of an outgoing edge at v iff in the polar dual, the rotation lowers the intersection of ℓ with the hyperplane spanned by the facet, see Figure 30.

In particular, local maxima at level k in the arrangement $\mathcal{A}$ correspond to k-facets of the polar dual point set that are entered by ℓ, see Figure 33. Thus, the exact version of Lovász' Lemma for point sets, Clarkson's bound on local k-level maxima, and McMullen's Upper Bound Theorem are mutually equivalent.

10.3.6. *The g-Theorem.* The possible integer vectors $(h_0, \ldots, h_d)$ that can appear as h-vectors of a simple d-dimensional convex polytope (the zeroth column of a bounded LP in general position) have been completely characterized in terms of the following conditions that together are necessary and sufficient. This characterization was conjectured by McMullen and proved by Stanley [**131**] (necessity) and Billera and Lee [**31**] (sufficiency). A simpler (but still rather involved) proof of the necessity part was given by McMullen [**104, 105**].

(1) *Dehn-Sommerville Equations*: $h_j = h_{d-j}$ for $0 \leq j \leq d$.
(2) *Generalized Lower Bound Theorem* (*GLBT*): If $g_j := h_j - h_{j-1}$ (with the convention $h_{-1} = 0$), then $g_j \geq 0$ for $j \leq d/2$.

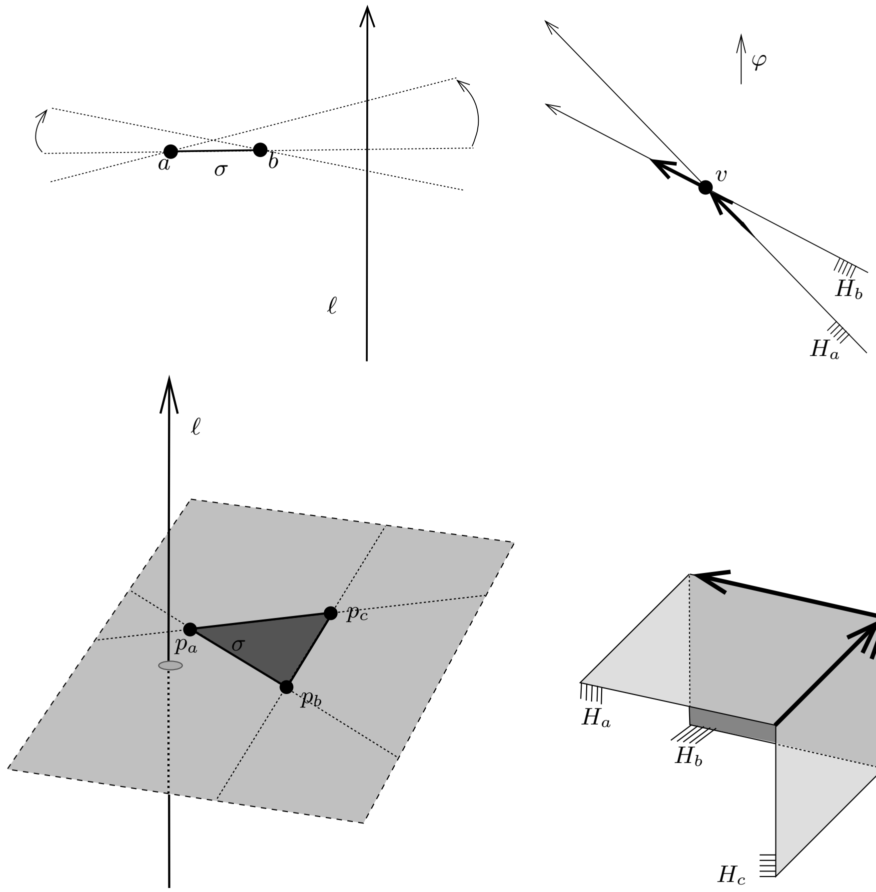


FIGURE 32. Incoming and outgoing edges under polar duality.

(3) Given positive integers a and r, there is a unique binomial expansion

$$a = \binom{a_r}{r} + \binom{a_{r-1}}{r-1} + \dots \binom{a_s}{s}$$

with $a_r > a_{r-1} > \dots > a_s \geq s \geq 1$ (one can choose the a_i's in a greedy fashion). Given this expansion, define

$$a^{\langle r \rangle} := \binom{a_r + 1}{r+1} + \binom{a_{r-1}+1}{r} + \dots \binom{a_s + 1}{s+1}$$

and set $0^{\langle r \rangle} := 0$. With this notation, the differences $g_j = h_j - h_{j-1}$ satisfy $g_{j+1} \leq g_j^{\langle j \rangle}$ for $0 \leq j \leq d/2 - 1$.

It seems too much to hope for a similar characterization of all possible h-matrices. However, there are a number of interesting open questions. The first one is how to define an h-matrix for arrangements of hemispheres, so as to avoid issues of (un)boundedness and the dependency on the linear objective function. A related question is whether one can find an algebraic interpretation for the h-matrix. A first step in the proof of the g-Theorem is the interpretation of the h-vector of a

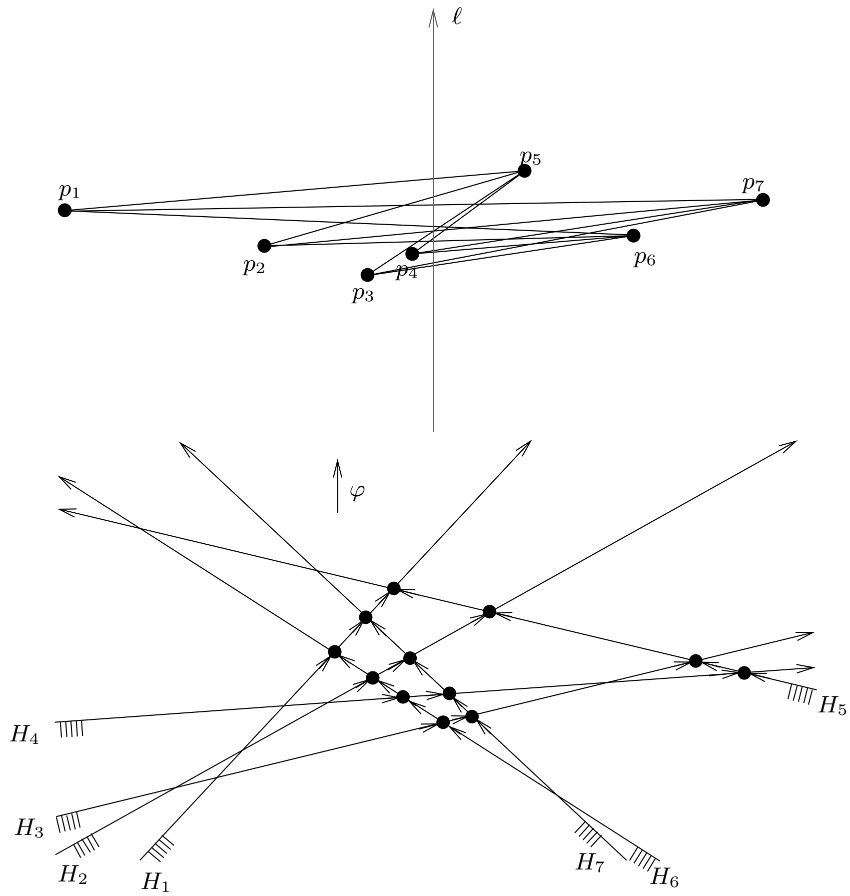


FIGURE 33. A line entering k-facets and local k-level maxima in the polar dual.

convex polytope as the Hilbert series of a certain graded algebra asscociated with the polytope, the *face ring* or *Stanley-Reisner ring* in Stanley's proof and the *weight algebra* in McMullen's proof. It would be interesting to define analogues of these algebraic objects for higher levels in arrangements.

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